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ON NON-SPHERICAL SURFACES IN OPTICAL INSTRUMENTS

by Allvar Gullstrand

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I - Method

The attention of researchers has been directed to the utilization of non-spherical surfaces in optical instruments since Kepler and Cartesius. Extensive practical use among the surfaces belonging to this category was first made of the cylindrical surfaces in eyeglasses for correction of ocular astigmatism. Now toric surfaces are also being used for this purpose and the wide distribution of astigmatism of the human eye has brought it about that eyeglasses with cylindrical or toric surfaces belong among the most occurring optical instruments.

However, if we restrict ourselves to non-spherical planes of rotation, it is no longer possible to speak of a general utilization. This doubtlessly is related to the difficulties both of the production as well as of the calculation of such surfaces. The present short history of the latter in practical optics is that of the surmounting of these difficulties and we can hardly go amiss by predicting that real progress in this field will be accomplished only by methods which permit both practical production with simple mechanical as well as calculations with elementary mathematical means.

Until recently, such surfaces seem to have served only for the purpose of focussing in one point of the axis. From this viewpoint, parabolic mirrors are highly suitable and have been utilized in telescopes, searchlights, and microscope condensers. This type presents no difficulties of calculation and the technical difficulties in production can be overcome as results have proved so that the only remaining question of importance is that of cost which does, however, influence the possibility of wide use. To an entirely different type belong the surfaces produced through so-called local retouching which have been employed principally for large telescope

objectives and where surfaces ground initially spherically were further processed by polishing of given zones. As a test of the effect of such retouching, it has been possible only to utilize the efficiency of the objective so that the method must be designated as purely empirical. The surfaces obtained in this matter are consequently unknown from the mathematical point of view and cannot be designed by calculation.

Only the introduction of a third type of non-spherical surface of revolution by Ernest Abbe[1] opened the path to the utilization of such surfaces for the correction of deviations of oblique [light] beams and only 10 years later was it possible for Rohr[2] to report on a practical success of this type by correcting the astigmatism of oblique bundles in cataract glasses. Abbe has embodied his respective findings in two patent descriptions which are dated 1899 according to Rohr[3]. In the first of these descriptions, he introduces, for the determination of a point on the surface, the distance of this point from the osculating sphere belonging to the vertex. This distance is designated by x and the radius of the osculating sphere by r^o. The distance s is calculated as positive when the point of the surface is located within the osculating sphere so that the radius vector from the center of curvature of the latter has the value $r^o - s$. The length of arc of meridian in the osculating sphere from the vertex to the point where it is intersected by the prolonged radius vector is designated

1. ERNST ABBE, Linsensystem mit Correction der Abweichungen schiefer Bueschel, Gesammelte Abhandlungen von Ernst Abbe, Vol II, P. 301, Jena 1906. --Verfahren, sphaeroidische Flaechen zu pruefen und Abweichangen von der forgeschriebenen Gestalt nach Lage und Groesse zu bestimmen, ibid, p.311.
2. M. v. ROHR, Uber Gullstrandsche Starbrillen mit besonderer Beruecksichtigung der Korrektion von post-operativem Astigmatismus. Report on the 36th meeting of the Ophthalmologischen Gesselschaft Heidelberg 1910, Wiesbaden 1911.
3. ibid, p. 189

by him with l and the equation of the meridian curve of the "spheroid surface" written as:

$$s = \frac{1}{4}kl^4 + \frac{1}{6}ml^6 + \frac{1}{8}nl^8 + \cdots.$$

By this designation, he intends to express the assumption that s is very small everywhere in relation to the length of radius r^o. Based on this assumption he developed approximate formulas which take into account only the first member of the series. The second patent description treats the method of production of spheroid surfaces by stating first that only such spheroid surfaces are suitable for optical purposes in which the linear deviation s from the sphere of the vertex remains restricted, within the effective area of the surface to a very small magnitude (a few 1/100 mm). The procedure consists in reducing, over given areas, the spherical surface by grinding and polishing and by employing an accurate lens as template so that it "does offer somewhat greater but no other difficulties than the representation of a spherical surface by means of the corresponding spherical lens template." These difficulties are predominant in the production of the template lens, and in general in the first production of a given spheroid surface because only spherical surfaces can be utilized for testing. In the preliminary processing of the lens surface, the coordinates of the surface point are checked by utilizing a suitable spherometer and compared with those of a model sphere with accurately known radius and with the same sign as that of the curvature of the vertex of the spheroid surface to be reduced and only little different from the radius. It is stated that the desired shape of the spheroid surface can be produced in this manner without difficulty in all zones with an accuracy of +/- 1 micron, provided the model sphere utilized is correspondingly accurate, but that an appreciably greater accuracy of the spheroid shape is required in general for optical purposes. In order to achieve this accuracy, a second model sphere is

utilized during the last retouching and the radius of curvature of this sphere with opposed sign is numerically approximately equal to that of the vertex of the spheroid surface to be tested and where the diameters of the interference rings, when placing this model sphere against the spheroid surface, represents the test object. If the radius of curvature decreases for a convex spheroid surface and/or increases for a concave surface toward the periphery, the two surfaces then touch at the vertex. If this is not the case, then the model sphere must lie on the spheroid surface along a strictly circular edge. The procedure makes it possible to produce accurately spheroid surfaces of given shape, "except for very small fractions of a micron." If the desired spheroid shape differs everywhere so little from the spherical shape that it can be produced from a sphere of suitably selected radius only by polishing over given areas, then the preparatory shaping by means of the spherometer can be eliminated.

The Abbe method is of such decisive importance that it requires more detailed discussion. Concerning first the practical results, the original consideration of Abbe seem to have been realized unchanged only in the already mentioned cataract glasses with any great degree of success because there are no other known optical instruments in which the divergence of oblique beams is corrected by surfaces whose shape is little different from the sphere. In any event nothing is known on whether any surfaces of this type have had any appreciable practical success. However, the Carl Zeiss Company has repeatedly used other non-spherical surfaces for the purposes of more accurate focussing in axial points, i.e., in large searchlights, in certain microscope condensers, etc. Rohr[1)] assumed from the beginning the

1. M. v ROHR, Ueber neuere Bestrebungen in der Konstruktion ophthalmologischer Instruments. Report on the 37th Meeting of the Ophthalmologischen Gesellschaft Heidelberg 1911, Wiesbaden 1912, p. 53.

task of calculating a single lens intended to give a sharp picture of an object with relatively wide beams. Calculation leads to the aplanatic lens in which the sinus condition is also complied with, at least for a given ray inclination. For certain delicate ophthalmological methods of examination, such lenses are absolutely required and have found wide distribution as components of the respective instruments but are also used for many other purposes. Although nothing further has been published on the production of such surfaces, Rohr seems to indicate that the procedure is based on the Abbe method. Theoretically also, very little has been published in this field by the followers of Abbe. The surface equation of Abbe is reiterated both by Koenig and Rohr[1] as well as by Siedentopf[2] but with changed designations, opposed sign and also (Siedentopf) changed numerical value of the coefficient. Koenig and Rohr derived the formulas which indicate the influence of Seidel image errors through the first coefficient in the series of Abbe. (In these formulas, the coefficient χ has the same sign and the same value as k above.) As far as nomenclature is concerned the designation of "spheroidal" selected by Abbe is not a happy choice because spheroid also encompasses an ellipsoid of rotation. In the first of the patent descriptions quoted above, there occurred the expression "spheroidal deformation of the original spherical surface" which is understandable in any event but has led to the unfortunate designation "deformed surface." However, in later publications of Rohr, the expression, non-spherical surface, is employed for a non-spherical surface of revolution, has been accepted elsewhere and is here used in this meaning.

Other than that, only few theoretical studies on non-spherical surfaces are available. In postulating the general problems by applying the

1. M. v. ROHR, Die Theorie der optischen Instrumente, Vol I, Berlin 1904, p. 323 ff.
2. ibid, p. 25.

iconic ["eikonal"] concept, both Schwarzschild[1] and Kohlschuetter[2] take into account the respective differential quotients in the equation of a general surface of revolution. Whereas these investigations were restricted to the paraxial space and consequently considered only the differential quotients of the surface equation at the vertex, I have deduced[3], on the exclusive assumption of a symmetrical plane, general formulas for calculation of the divergences of finite inclined beams from which result, for general axial-symmetrical systems, some of the asymmetrical values determining the decimal points, the inclinations of the two image surfaces and the asymmetrical value of the tangential enlargement coefficient as well as certain other summation formulas[4] which can be used in the calculation of optical systems.

The method of differential geometry employed in these investigations differs basically (because of the difference of the task) from the treatment of the problem to comply with given conditions for each ray in a wide-open system. Schwarzschild[5] calculated a system consisting of two mirrors and strictly aplanatic for infinite distance from objective in which not only the axial image point is strictly free of aberration but which also fulfills the sinus condition along each ray, and was able to represent the rectangular coordinates of the surface points for both mirrors without approximation as explicit functions of a parameter. The same problem, even

1. K. SCHWARZSCHILD, Untersuchungen zur geometrischen Optik. Abhandlungen der Gesellschaft der Wissenschaften zu Goettigen, Math. phys. Klasse, new series, Vol IV, N:o 1-3, Berlin 1905.
2. ARNOLD KOHLSCHUETTER, Die Bildfehler fuenfter Ordnung optischer Systeme abgeleitet auf Grund des Eikonalbegriffes nebst Anwendung auf ein astrophotographisches Objektiv. Diss. Goettingen 1908.
3. Die reelle optische Abbildung. This series, Vol 41, No. 3, Upsala 1906
4. Tatsachen und Fiktionen in der lehre von der optischen Abbildung. Archiv fuer Optik, Vol 1, 1908
5. ibid., No. 2.

complemented by a requirement for achromatism, was solved by Linnemann[1] for refracting surfaces. However, the surface equations can here not be simply written down but the respective differential equations must be integrated numerically. In connection with such integration, there is also indicated the numerical method with which a ray can be followed through such a system.

Let us mention finally that the greater interest in non-spherical surfaces is manifested by the fact Kerber[2] outlines the application of his new calculation formulas for deformed rays on non-spherical surfaces and Lange derives both the calculation formulas for rays traveling in a meridian plane[3] as well as the aberration values in the axis for surfaces of revolution of the second order[4].

This brief review shows the non-spherical surfaces utilized so far in practice are either surfaces of revolution of the second order or Abbe surfaces by which I understand surfaces of rotation whose meridian sections have the equation indicated above. We here disregard the mathematically non-definable surfaces produced empirically by local retouching. In addition, theoretical treatment has been given to reflecting surfaces which are represented by certain transcendant equations and certain surfaces suitable only for numerical integration of differential equations. In this connection, we want to mention also, among earlier investigations, the Huygen method[5] with which it is possible to construct a surface

1. MARTIN LINNEMANN, Ueber nichtsphaerische Objektive. Diss. Goettingen 1905.
2. ARTHUR KERBER, Neue Durchrechnungsformeln fuer windschiefe Strahlen. Zeitschrift fuer Instrumentenkunde. Vol. 33, p 75, 1913.
3. MAX LANGE, Durchrechnungsformeln fuer die Lichtbrechung an Kegelschnitten. ibid. Vol 34, p 273, 1914.
4. The same, Entwicklung des ersten Gliedes der Aberration endlich geoeffneter Lichtbueschel fuer den Achsenobjektpunkt einer lichtbrechenden Rotationsflaeche deren Querschnitt ein Kegelschnitt ist. Ibid. Vol 31, p. 348, 1911.
5. CHRISTIAN HUYGENS, Treatise on Light. Rendered into English by SILVANUS P. THOMPSON. London 1912, p 116 ff.

point-by-point and which makes a given light beam homocentric by refraction.

Concerning these last categories, it will be evident that the respective surfaces can be produced equally well with the Abbe method as the Abbe or any other surfaces. However, it is equally evident that they require much more complicated calculation and are therefore replaced, in practice, by the last named or other surfaces equivalent in this respect. This is probably also always possible. Actually, we are not concerned in the first instance with a fusion of the rays in a mathematical point since diffraction produces the fact that, even in those cases where such fusion of rays is accomplished theoretically, a finite extended area takes the place of the mathematical point, and in the second instance, it is possible to select a surface by including an adequate number of coefficients in the Abbe series so that it will osculate a given surface with the desired accuracy and will still be maniable for mathematical calculation. In practice, we should therefore be able to confine ourselves for the time being to calculations with surfaces of the second order and with Abbe surfaces and the two categories which complement each other, should be adequate for the requirements concerning non-spherical surfaces in optical instruments.

We might be tempted to conclude from this that all reasonable objectives in regard to such surfaces have already been achieved or are at least easily achievable. This would perhaps be the case if the question of the production of such lenses were not of such decisive importance. At the present time, the problems of geometrical optics are being investigated in the scientific world by very few researchers who are not connected in one way or another with the optical industry. That this relation has not had only an immense progress of technical optics as consequence but has

also appreciably advanced pure geometric optics, we can only gratefully confirm in regard to the latter. However, on the other hand, this relation has also disadvantages for science because certain advances are not always published but are treated more or less as industrial secrets. It is obvious that we cannot reproach the respective inventors or discoverers for this because otherwise there would certainly have been achieved much less success and geometric optics would consequently have been less advanced. However, this situation results in regard to the production of non-spherical surfaces in the fact that information can be obtained only from patent descriptions and generally in a very restricted form.

As far as the Abbe method is concerned specifically, the latter can be utilized unchanged only for surfaces which deviate very little from the spherical form whereas this deviation assumes considerable proportions, for example, in the aplanatic lenses of Zeiss. If the surfaces are produced by a method based on the Abbe theory, the latter must have undergone certain modifications. Since nothing has become known of this, however, we can only attempt to examine the manner in which this method can be modified for such purposes.

In a simple lens with only one non-spherical surface, and when absolute absence of aberration in one axial point is required, the sinus condition can be complied with in general only for a general beam inclination. This is achieved easily by an appropriate deflection [curvature?] of the lens whereas, for a given deflection, the coordinates of the points on the non-spherical surface can be determined in any desired number and desired accuracy through the Huygens method. In order to do so, it is only necessary to express the geometrical expression of Huygens trigonometrically. As already indicated, this results in two possibilities for the production

of the lens template by either producing this surface directly or else calculating a surface by means of the Abbe equation which will adequately closely osculate the former to be utilized in production. In the first case, there must be calculated at least as many point coordinates as are required for spherometric control and, if further control by means of the interference rings is to be used, very complicated calculations must be added in order to achieve the accuracy intended by this method and without which it would seem rather useless. However, for the calculations of the Abbe surface, the coordinates of a small number of surface points are sufficient. The calculation by which they are determined, also furnishes the direction of the normal and the radius of curvature and it is possible -- as will be explained in detail further below -- to calculate the osculating surface in various ways, even by utilizing only one single surface point but where then the residual amount of aberration of different rays must be investigated by theoretical calculation in order to judge whether the calculated surface osculates the prescribed shape sufficiently accurately. If the surface equation has been found in this manner, the calculations necessary for using the interference rings for control can be effected more easily than when the surface can be constructed only point-by-point. As already stressed, since we are actually not concerned with a mathematically accurate fusion of rays, there seems to be no reason to prefer the more difficult calculation of a surface constructed point-by-point to the Abbe surface calculated with an adequate number of coefficients.

The method of control by means of the interference rings cannot be employed unchanged when a high deviation from the spherical shape exists. A surface of revolution can be regarded as the enclosing surface of spheres whose centers are located in the axis and whose radii are equal to the

normals drawn from the respective axial point to the surface. If then the radius of curvature steadily increases from the vertex to the periphery and the evolute has an apex in the direction of the vertex, then the distance of the center of the sphere from the vertex as well as the radius of curvature of the meridian section of the surface is greater everywhere in the points touched by the generating sphere than the radius of the latter and the sphere is located on the concave side of the surface without intersecting the latter. Precisely the opposite takes place if the radius of the curvature of the surface steadily decreases from the vertex to the periphery when the generated surface always lies within the generating sphere without being intersected by the latter. It follows from this that, in the production of a lens template, control by means of the interference rings created in the contact with spherical surfaces along the latitudes of the ground surface is possible only if we grind, for an increasing radius of curvature (from the vertex to the periphery), a concave and, for a decreasing radius of curvature, a convex lens template. Depending on the degree of deviation from the spherical form and depending on the required accuracy, the check must then be carried out by means of a greater or lesser number of different spherical surfaces. Instead of contact, we obtain osculation at a very small angle by grinding an appropriate circular edge on the spherical surface. A given latitude or parallel of the ground surface corresponds to each radius of sphere. If we now imagine the spherical surface divided into two parts by section in such manner that the plane surface of section has exactly the diameter of the corresponding parallel and if we imagine further that this surface of section is ground off somewhat but only very little in both parts, then each of the two parts can be placed against the ground surface so that the edge

of the spherical surface coincides with a parallel of the ground surface and the two surfaces here intersect at a very small angle. Other possibilities also exist on which we shall not enter here since we are merely concerned with showing that the Abbe method of control by interference rings can be modified so that the method can be employed also for surfaces whose form differs appreciably from the spherical form. However, it is obvious that the method becomes complicated in proportion to the increasing degree of deviation and requires a corresponding greater number of spherical lens surfaces with accurately known radius.

For the first shaping of surfaces differing appreciably from the spherical form, a shorter procedure is preferable to subsequent zonal grinding and polishing. Preliminary grinding by machine is highly suitable for this, since grinding is based in general on the contact of two surfaces and a non-spherical surface to be produced by employing one single grinding surface generally cannot touch the latter simultaneously in more than one point so that the machine must obviously be capable to accomplish such movement of the two surfaces in relation to each other that the non-spherical encompasses the grinding surface. It is of technical advantage here when both surfaces move in themselves and when we start from a suitably selected and spherically ground lens. The latter is consequently centered on a rotating axis and the grinding surface best represents a surface of revolution which rotates around its own axis. If this surface is neither cylindrical nor degenerates into a sphere or a plane, we can then grind only through a single parallel of the grinding surface because the mechanical installation would otherwise become very complicated. This parallel latitude must then always be located in a plane which contains also the axis of revolution of the non-spherical surface and the grinding

surface must have such a form that no other point of it can touch the non-spherical surface. Although in general suitably formed surfaces of revolution can be employed as grinding surfaces which are symmetrical to an equatorial plane and such symmetry is not even necessary, it would seem to be sufficiently generalized for the present demonstration if we proceed on the assumption that the grinding surface represents a torus and that both the cylinder as well as the sphere and the plane are considered as special cases of the torus. The toric surface may be regarded as the encompassing surface of a sphere whose center moves on a fixed circle, the base circle and its axis of revolution therefore intersects the plane of the base circle perpendicularly in the center of the latter.

The requirement for having the grinding parallel and the axis of revolution of the non-spherical surface always located in one and the same plane is most easily complied with by having both the axis of revolution as well as the base surface of the torus always remaining in one and the same fixed plane. If we consider the axis of revolution as fixed, the center point of the toric surface must therefore describe a fixed curve which represents a curve parallel to the meridian curve of the non-spherical surface and will be designated as machine curve. Since the base circle of the toric surface must remain in the fixed plane and the axis of revolution of the latter therefore must always be perpendicular to this plane, the machine curve is consequently described by each point on this axis. However, if we grind with a sphere, the orientation of the axis of revolution becomes indifferent and the machine curve needs to be described only by the center of the sphere. In this manner, it is possible to grind a convex surface with a concave spherical calotte. On the other hand, if a convex surface is to be ground and a cylinder is substituted for the

toric surface, then a displacement of the cylinder in the direction of its axis is permissible although this axis must then continuously intersect the fixed plane perpendicular in all positions. If such displacement takes place, the machine curve is then described by the point of intersection of the cylinder axis and the fixed plane. Finally, if the non-spherical surface has no points of inflection on the meridian curve, the cylinder can be replaced by a plane which is articulated around an axis lying in or parallel to this plane and intersecting the fixed plane always perpendicularly. The inclination of the latter in the different positions of the axis of articulation is then determined thus that the line drawn in the fixed plane from a fixed point located on the axis of revolution of the non-spherical surface to the intersection with the axis of articulation always represents a normal of the grinding surface. This plane can then perform any desired movement in itself and the machine curve described by every point on the axis of articulation represents, depending on whether this axis lies or does not lie in the plane, the base curve of the meridian curve of the non-spherical surface or the base curve of a parallel curve of the latter in relation to the fixed point. Whichever of these arrangements is selected, the machine curve now can always be constructed point-by-point without difficulty as soon as the equation of the non-spherical surface is given and we need in essence only to describe only a curve constructed point-by-point by a machine part.

This problem cannot be solved purely kinematically but requires employment of methods which may be grouped under the designation, template method, and are characterized by the utilization of a guiding curve constructed punctually. This curve need not be similar to the machine curve but can be produced by any kinematic generation of one curve out of another,

i.e., by circular inversion, by rolling up or off, etc., so that eccentric curves and evolutes are also included among the guiding curves in connection with it. By employing a guide curve not similar to the machine curve, it is also possible to make the errors of the machine curve smaller than those of the guide curve. This could be accomplished by pantographic transfer in which the guide curve would be similar to the machine curve but be constructed on a larger scale. Template methods seem to have been employed already for a considerable time[1].

A particular method of preliminary shaping has been patented by Carl Zeiss[2]. A lens first ground with two spherical surfaces is softened by heating so that it adapts itself to a supporting surface also ground spherically. The process imparts to the upper surface of the lens a non-spherical form depending on the radii of curvature of the three spherical surfaces and on the thickness of the lens. Whether this method has proved itself is not known. In any event, the risk of internal stresses in the lens represents a complication which cannot be disregarded[3].

It should be clear from the foregoing that the original Abbe method, in order to be applicable to non-spherical surfaces whose shape differs appreciably from the spherical form, needs to be modified only so that preliminary shaping is effected by means of a template method (or possibly by heating), that more coefficients are included in the equation, and that a larger number of spherical surfaces with accurately known radius of curvature

1. See, eg., the German patent No. 23369, E. Avril.Schleifmaschine um Brillenglaeser nach einem Modell zu schleifen. Zeitschr. fuer Instrumentenkunde, vol 4, p. 74, 1884.
2. D. R. P. No. 212621. Deutsche Mechaniker Zeitung 1910, p.51.
3. See the communication of Schott et al. Der Einfluss der Abkuehlung auf das optische Verhalten des Glases und die Herstellung gepresster Linsen in gut gekuehltem Zustande. Zeitschr. fuer Instrumentenkunde, Vol 10, S 41, 1890.

is employed for control with the interference rings.

We do see on the other hand that this method is too complicated for producing a large selection of non-spherical surfaces but can be employed for making a small number of such surfaces which can be used in a sufficient number for given purposes. To this should be added that this selection is determined not only by the shape of the surface but also by the radius of curvature of the vertex. For example, if the aplanatic projection of a point is desired by a simple lens of given focal distance and corresponding lenses with this focal distance are not available, there is then required, for the production of the desired lens, a new lens template in any event and possibly also a new guide curve. The latter is eliminated in two cases; first, when the method of preliminary shaping by heating is possible in practice and, second, when the preliminary grinding machine is constructed so that the guide curve determines only the shape of the surface but not the ratio of magnitude. It is obvious that the complicated production and consequent restriction in the selection of surfaces produced is a handicap in the general use of the latter in optical instruments.

However, there is no doubt that a wide field is open to non-spherical surfaces as soon as the production of the latter is possible under such conditions that the designer of optical equipment can incorporate non-spherical surfaces without any hesitancy in his design.

Two objectives here occupy the foregoing. In order to advance essentially the possibilities of construction in general, it would be desirable that there should always be available surfaces of a selected simple type in which evidentally, aside from the radius of curvature of the vertex, only one coefficient could be freely available. If it were always possible to employ only a single such surface in an optical instrument and even if

this possibility were restricted only to convex non-spherical surfaces, it will be easy to see that even this would accomplish an appreciable extension of the optical means available. We need only remember that, in general, a Seidel image error can be corrected through the respective coefficient. The second objective must obviously be to encourage the utilization of such surfaces which will osculate, with optimum possible accuracy, a surface of any desired prescribed form. The means for this would be a simplified production of surfaces of an appropriate type with several coefficients.

The possibility of simplified production of non-spherical surfaces is influenced to a great extent by the requirements made on the accuracy of the shape of the surface. In respect to this, relatively high demands for axial symmetry must be fulfilled whereas small deviations in the shape of the meridian curve are more easily permissible. If the surface does not represent strictly a surface of revolution, this then results in deviations on the axis which are most closely similar to those which are created by inadequate centering of an optical instrument and are as little permissible as these. However, if the shape of the surface to be produced deviates appreciably from the spherical form and we obtain a surface, in place of the desired surface, which, although it does represent a surface of revolution, has a meridian curve with minor zonal deviations from the prescribed form, these will be, in the greater part of the cases, without significance for the practical application of the respective optical instruments. This can be best illustrated by an example. The hyperboloid of rotation whose meridian curve has a numerical eccentricity which is equal to the value of the refraction index of the lens, converges -- as is well known -- a beam of rays of arbitrary width which is parallel in the glass medium, on

one point in the air medium. To begin with, it is obvious that any deviations of the meridian curve of the surface from the prescribed hyperbolic form which cause no greater divergences of the rays than are still contained within the diffraction disk, are completely without significance. However, even zones of greater deviation are generally permissible without any hesitancy. Such zones do occur in existing optical instruments very frequently without interferring with the usefulness of the latter. Even if no single point of the surface should lie exactly on the prescribed hyperbola, such a surface, provided that it strictly represents a surface of revolution and that the meridian curve is free of any unsteadiness and has approximately the prescribed hyperbolic form, would constitute an enormous step forward from the spherical surface. It follows from this that, although we must require a strictly axial-symmetric form and cannot permit unsteadiness on the meridian curve, there is no reason for making excessively high demands on the trace of the meridian curve, unless we are concerned with surfaces whose shape differs only very little from the spherical form. The degree to which the demands may be reduced can only be decided by success in practice. Initially, the only path open is then to examine the possibility of a simplified production of non-spherical surfaces from the point of view of this principle.

If we proceed to this from the modification of the original Abbe method outlined just now, it would appear initially that the complicated control of the lens template by means of the interference rings can be eliminated because the spherometer method seems sufficiently accurate, even though the latter possesses only an approximate accuracy, in relation to that indicated by Abbe, for surfaces whose shape varies appreciably from the spherical form. However, we may well ask whether a lens template is absolutely

necessary. A thorough simplification of production should be oriented toward mechanical working methods as much as possible in view of industrial operations. In the respective machine, the surface must be capable of being produced so accurately that no further finishing except perhaps polishing is necessary. The latter can then be effected by skilled workers without any essential change in the form of the surface. To what extent optical or other control methods will here be required can be determined only from experience. Where surfaces are concerned of which a large number are to be produced, the lens template will of course again take its proper place.

When such requirements are established, it is obvious that production by machine cannot be based on a template method because the production of the curved-line guidance would require such complicated working that the intended simplification might become questionable. The respective machine curves must therefore be capable of being produced kinematically and only such guide curves should be employed which can be produced exclusively by machine. It follows from this that the meridian curve of the non-spherical surface cannot represent an Abbe curve. On the other hand, since the machine curve cannot coincide with this meridian curve because this would require grinding with one cusp ["Spitze"] represents a parallel or base curve of the latter, it is evident that the equation of the meridian curve of the non-spherical surface in general will not be obtained at all. The resulting disadvantage that we must start from the machine curve in the calculation of optical systems which contain such non-spherical surfaces, is of completely minor significance since the additional calculations -- as will be explained in further detail below -- demand only trigonometric methods. However, exceptions to this are all those cases in which it is necessary

to determine the intersection of the non-spherical surface with a given ray but in these cases we are restricted to numerical methods even when employing the Abbe equation.

The disadvantage of having to employ the machine curve in place of the meridian curve in the theoretical calculation of the system can be eliminated only when the latter curve possesses a parallel or base curve which can be generated kinematically and can itself be expressed directly through an equation. Theoretically, this is the case for all curves whose parallel or base curves represent algebraic curves since any plane algebraic curve can be generated by a system of articulation. However, since the machine will be as much more accurate as it is simple, only a very few known curves can be utilized in practice and we cannot expect in this manner to be able to generate curves with more than one freely available coefficient. On the other hand, we have already stressed that such curves can be expected to play an important role. Among them, curves of the second order are at the top, both because the latter possess base curves which can be easily generated kinematically and also because of the possibility to carry out theoretical calculation in all cases by employing the most elementary means. Since the machine curve represents a base curve, only convex surfaces can be obtained with such direct methods but the advantage of the simpler calculation is so great that the surfaces of revolution of the second order generally do seem to be most suitable to comply with the purpose indicated above, provided that we are successful in producing the latter with adequate accuracy and with the numerical eccentricity which can be arbitrarily prescribed within sufficiently large limits at any desired radius of curvature of the vertex.

In the solution of the other task, i.e., to produce non-spherical

surfaces which will osculate sufficiently accurately a prescribed surface, emphasis must be placed on the possibility of calculating the respective coefficients in the equation of the machine curve which we shall designate as machine constant, for the sake of brevity, with mathematical means as simple as possible. Above all, the equation of the machine curve must therefore be suitable for an osculation of higher order in the vertical point. Since it is desirable to have available as large as possible a number of machine constants for this purpose and the respective calculations for more than three constants are too complicated, we shall concentrate on the problem of obtaining a complete contact of the eighth order, as far as possible, of the ground surface with the prescribed surface, i.e., a nine-point contact of the meridian curves of the two surfaces in the vertical point. For this purpose, we must above all determine the respective generally valid relations between the differential quotients of the curve and those of the parallel or base curve for a vertical point. If we have found the machine constants which will produce such a central osculation of a given order, it is best to employ the latter not always unchanged but to attempt a higher degree of osculation through variations of the latter under control by theoretical calculation or by the values prescribed for the surface. With very large openings, it may be of advantage to base the calculation of the machine constant on the condition that the meridian curve of the ground and of the prescribed surface shall touch each other in a given point or shall possibly also have the same radii of curvature. For such an eccentric osculation of the first and/or second order, two and/or three machine constants are required. In order to obtain simultaneously either a central osculation of the fourth order or an eccentric osculation of the second or two eccentric osculations of

the first order, four machine constants are required. For the Abbe curve which seems specifically adapted to this purpose, the coefficients in such conditions can be determined by linear equations. We are therefore confronted by the task to find a machine curve equally ideal in this respect and to have in mind, in any event, in the selection of the machine curve the possibility of specifying various conditions in the determination of the machine constant.

This investigation of the various methods for calculating the machine constant will be followed by a demonstration of the methods for the theoretical calculation of optical systems containing the respective surfaces and followed ultimately by a discussion of the employment of such surfaces for certain purposes.

II - New Methods of Production

Convex Surfaces of Rotation of the Second Order. The base curve of a conical section in relation to a focus is, as is well known, a circle which has its center in the center of the curve, whose radius is equal to the semi-axis and/or the major semi-axis, and which therefore degenerates, in the case of a parabola, into the vertical tangent of the latter. This is equivalent to the fact that the lines drawn through the different points of a circle perpendicularly on the connecting lines of these points with a given point, are encompassed by a conical section. It follows from this that, by employing circular and straight-line guidance, we are able to grind convex surfaces of revolution of the second order with one plane by the method outlined above. In Fig. 1, let AB be a crank which is able to

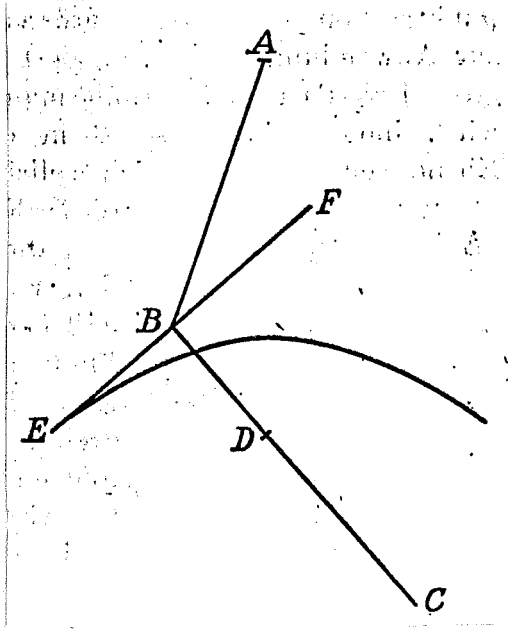

Fig. 1

revolve around an axis intersecting the paper plane perpendicularly in A and possessing an axis parallel to the axis in B around which the arm BC is able to revolve in turn. It is then merely necessary to allow this arm in any position to pass through the fixed point D and to rigidly link the grinding surface which intersects the paper plane in the figure

perpendicularly to the line EF, in such manner that it is located perpendicularly on the line BD and that the axis B is contained in it. When pivoting the crank arm, the plane EF is encompassed by a straight hyperbolic cylinder in which A represents the center point, AB the semiaxis and D a focus of the hyperbola. The axis of rotation of the lens surface to be ground must therefore be parallel to the paper plane and contained in a plane which intersects the latter perpendicularly in the line AD and grinding is effected with the plane grinding downward while the lens is raised upward with unchanged axis of rotation. If the plane EF intersects the line BD or its prolongation perpendicularly in any other point than B, the corresponding parallel surface of the hyperboloid will be ground.

The convex surface thus ground on the upper face is produced by pivoting the arm AB so that the axis B comes to lie between A and D in the center position. By a full revolution of the crank, however, the two branches of the hyperbola are generated. If the arm AB is then pivoted around a center position in which the axis B intersects the extension of the line AD so that the plane EF now grinds upward, we then obtain, on the same axis of revolution, the same hyperboloid which is now convex on the under face and/or the corresponding parallel surface. This is equivalent to the fact that the line DC in Fig. 1 is not permitted to run through the point E but through that point which is located at the same distance from A as B along the extension of the line AD so that then the lower shell of the hyperboloid is ground by employing the upper focus. As will be seen immediately, one and the same angle of traverse of the arm AB corresponds in this case to a lesser inclination of the plane EF toward the horizontal. To this should be added that the contact point of the plane with the surface is not located on the same side of the line AD as the axis B (Fig. 1)

but on the opposite side. For one and the same surface points, both the angle of traverse of the arm AB as well as the distance of the contact point from the axis B is greater when we employ the more distant instead of the nearer focus.

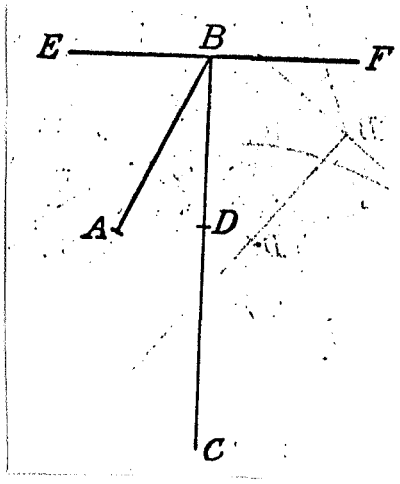

Fig. 2

The paraboloid is obtained when the crank is replaced by a straight-line guidance and if the axis B is guided perpendicular to itself in a plane located perpendicularly to the axis of revolution. The distance of the point D from this plane is one-half of the radius of the vertex and, if the plane EF intersects the line BD in any other point than B, there then results in the same manner the corresponding parallel surface. On the other hand, if the crank axis A is located below the point D and if AB is greater than AD, then the ground surface is an ellipsoid which has the greater axis as axis of revolution and/or the corresponding parallel surface of such an ellipsoid. By a full revolution of the crank, the complete ellipse is generated as the encompassing curve of the line EF. It follows from this that it will be possible also to grind ellipsoids of rotation with the shorter axis as axis of revolution. In Fig. 2, if A is the fixed axis, AB the crank arm, BC the line passing in any position always through the fixed point D, and EF the secant of the grinding surface located perpendicularly on the line BC and rigidly linked to the latter, there will then

be ground, when pivoting the arm AB around the center position drawn in the figure, such an ellipsoid and/or the corresponding parallel surface of the latter, under corresponding securing of the grinding surface, if the axis of revolution of the surface is located parallel to the paper plane and contained in the plane perpendicular to AD which passes through the point A. If the arm AB is pivoted only to one side, it will be easily seen that, for one and the same surface point, both the angle of traverse as well as the distance of the contact point of the grinding plane with the ellipsoid of axis B is smaller when AB is brought closer to D during pivoting than in the obverse case. Simple reflection will show us that the same differences exist also in grinding an ellipsoid with the greater axis as axis of revolution by utilizing the closer and/or the more distant focus.

Mechanically, the requirement for having the line BC pass in all positions through the point D is complied with by having the arm BC slide in a straight-line guide which can rotate at D around an axis intersecting the paper plane perpendicularly and having the direction of the line BD coincide with that of the straight-line guide. The mechanism is consequently an oscillating crank mechanism or, in the more exact nomenclature introduced by Burmester[1)], a centrally rotating and/or a centrally oscillating crank-drive depending on whether an ellipsoid or a hyperboloid is to be ground. If the crank is replaced by a straight-line guide for the purpose of grinding a paraboloid, this then results in a central straight-line thrust-drive, according to the same nomenclature. Translation into practice initially raises the question whether the arrangement outlined at the top of Fig. 1 in which the axes A and D as well as the

1. L. BURMESTER, Lehrbuch der Kinematik I. Leipzig 1888.

axis of revolution of the lens surface are fixed, is actually the most advantageous. If these components are not fixed, the axes A and D must be linked to each other by an arm and the axis of revolution of the lens surface must be fastened to this arm in such manner that it intersects the two axes perpendicularly. The mechanism then consists of the two arms AB and AD with the articulated axes AB and AD and of two equivalent components sliding against each other of which one can revolve around B and the other around D, and which provide straight-line guidance in the respective direction BD. These components, called in kinematics infinite members of the specialized plane mechanism, will be designated here simply as thrust-components. It is therefore a general condition for the grinding of the respective surfaces that the grinding surface is linked perpendicularly to the direction of thrust on one of the thrust components whereas the axis of revolution of the lens surface is rigidly linked to the opposite arm and intersects the articulated axes of the latter perpendicularly. Depending on whether the grinding surface passes through the articulated axis of the respective thrust component or not, a surface of the second order and/or the corresponding parallel surface is ground. Since AD represents the distance of a focus from the center point, AB the semi-axis and the ratio AD to AB is therefore indicated by the numerical eccentricity, the numerical eccentricity is therefore in general equal to the ratio of the length of the arm linked to the axis of revolution to the length of the other arm.

However, it is possible to fix rigidly any one of the four links and thus distribute the motion in different ways to the grinding surface and the axis of revolution. In Fig. 1, the axis of revolution is fixed whereas the plane carries out a compound motion. However, if we make the arm AD fixed -- which again results in a crank drive -- then both the axis of revolution as well as the plane effect simple motions of revolution, i.e.,

around the axes A and/or B. On the other hand, if one of the thrust components is made fixed -- which produces a thrust-crank drive -- either the grinding surface remains fixed while the axis of revolution carries out a compound motion or else the plane is slid back and forth in the direction of its normal whereas the motion of the axis of revolution is a simple rotation, depending on whether the thrust component containing the axis B or that containing the axis D is made fixed.

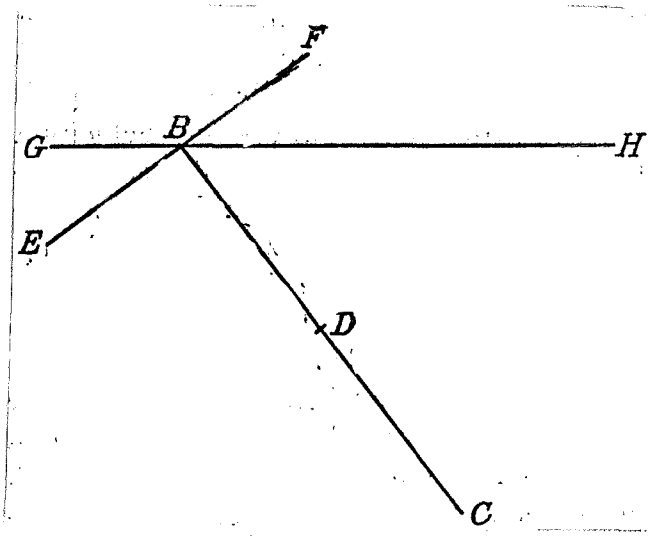

Fig. 3

Except for differences of mechanical design, the same is true for the sliding mechanism for the grinding of paraboloids. In Fig. 3, GH represents the straight-line trace of the point B which originates when the axis A in Fig. 1 is infinitely distant. The mechanism consists of two straight-line guides with two articulated axes and has four links of which one has an articulated connection with another one and slides in a straight line along a third. The connecting line of the axes B and D coincides with the direction of one of the straight-line guides. A central straight-line guide is therefore linked to an eccentric straight-line guide by two articulated axes. It will then be easily seen that, in this mechanism also, two links belonging to one and the same straight-line guide are kinematically equivalent.

This is due to the fact that, if we make the link connected to the axis B and belonging to the eccentric straight-line guide fixed, the axis D must then describe a path parallel to the line GH and if we then make the axis of revolution of the lens surface fixed to the fixed link and the grinding surface is made fixed to the link connected with the axis D and belonging to the central straight-line guide, we then have identically the same mechanism. It is consequently a general condition for grinding that the grinding surface must be made fixed perpendicularly to the direction of thrust on one of the links belonging to the central straight-line guide whereas the axis of revolution of the lens surface is connected to the opposite link belonging to the eccentric straight-line guide in such manner that it intersects the articulated axis belonging to the latter perpendicularly and is located perpendicularly to the direction of the straight-line guide. In the arrangement shown in Fig. 3, the axis of revolution of the lens surface is made fixed whereas the grinding surface executes a compound motion. If the thrust component containing the axis B and belonging to the eccentric straight-line guide is made fixed, then the axis of revolution executes a straight-line and the plane a rotating motion. On the other hand, we can make the plane fixed so that the axis of revolution executes a compound motion and if we then finally make fixed the link containing the axis D and belonging to the central straight-line guide, the plane then executes a straight-line motion in the direction of its normal whereas the axis of revolution rotates around the articulated axis D.

For the selection among these various types of machines, the guiding viewpoint must be to obtain optimum accuracy of motion and optimum exclusion of vibrations. It follows from this initially that sliding friction should not exist in straight-line guidance. Since this requires a carriage,

there remain only two types if we want to exploit the mechanical advantage of a fixed carriage path. If we consider further that the grinding surface must perform motion in itself which is accomplished most simply by rotation around a normal, we then obtain one each axis of revolution in two different links of the mechanism and the requirement for having one of these axes fixed simultaneously with the path of carriage travel is complied with only by that type in which the grinding surface is made fixed. For other reasons also, this type appears to be the mechanically most advantageous, at least where we are concerned with optimum applicability for the production of different surfaces in a small number of pieces. On the assumption that we want to grind a convex hyperboloid on the underface, this type is shown in the diagram of Fig. 4. The vertical axis seen on the left has fixed bearings and carries the horizontal grinding disk in which the plane grinding upward

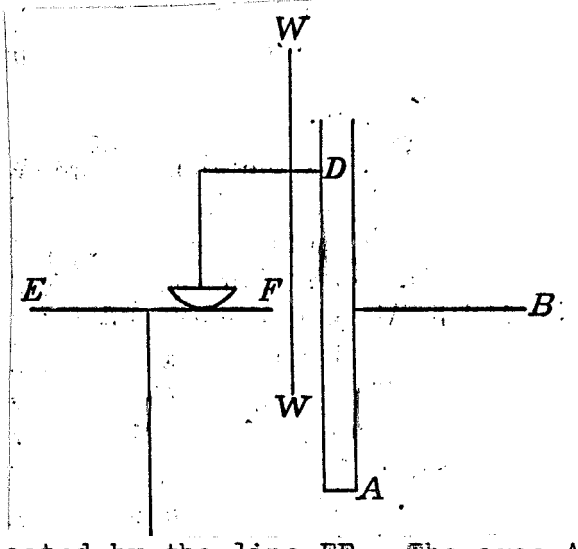

Fig. 4.

is indicated by the line EF. The axes A, B, D have the same significance as in Fig. 1. However, B is here the crank axis set in fixed bearings whereas A represents only the articulated connection of the crank arm with the couple. The other end of the couple is made fast to the axis D and the bearings of the latter are carried by the carriage W which moves vertically in a fixed path. On the left, the machine component carrying the axis of

revolution of the lens is linked rigidly to the axis D. If the crank is rotated, then the axis of revolution of the lens follows the motion of the couple. In order to adjust the machine for different surfaces, it is necessary to be able to give the crank arm and the couple the corresponding length. They must therefore be able to be made fixed to the respective axes at any desired point. Consequently, the distance of the lens vertex from the axis D must also be capable of being varied correspondingly and the path of the carriage must be sufficiently long in order to make possible guidance at the different initial positions of the axis D. Lack of space prevents us from going into further details. However, it is clear that the three axes must be capable of being very closely approached to each other when grinding surfaces with a small radius of vertex and a value of eccentricity either low or close to 1 and can therefore not be arranged above each other but must be next to each other as indicated in the figure. For purposes of adjustment, it is also advantageous to bring them into one line which requires them to be centered on each other. When grinding an ellipsoid, the axis A must be located above the axis D so that the full length of the crank arm, as drawn in the figure, can be made available.

For the grinding of a paraboloid, the crank arm and the couple must be removed and a roller fixed to the axis B. On this roller then rests a plane horizontal in the initial position which is made fixed to the axis D in such manner that the shortest vertical distance of the two axes from each other is equal to one-half of the prescribed semi-parameter. The same arrangement can also be applied to the grinding of ellipsoids and hyperboloids when the semi-axis is so large that the corresponding length of the crank arm would tend to result in mechanical failure. The plane is then replaced by a cylinder surface whose axis assumes the prescribed position of the axis A.

Since the crank arm is replaced in these cases by a "closed linkage", the force effective in pivoting must consequently have its point of attack on the couple. This is moreover of further advantage if the latter is shorter than the crank arm which is the case in the grinding of ellipsoids.

A machine of similar design can also be utilized for the grinding of ellipsoids which have the shorter axis as axis of revolution. It is merely necessary for this that, in the initial position outlined in Fig. 4, the axis of revolution of the lens can be made fixed to the axis D perpendicularly to the paper plane. The already postulated possibility of varying the distance of the lens vertex from the axis D in any of usual applications now serves for displacing the axis of revolution perpendicularly to itself until it is intersected by the extension of the axis A located above D. In addition to this, the lens vertex must also be capable of being displaced in the new position in the direction of the axis of revolution so that the distance of the latter from the axis A can be made equal to the shorter semi-axis of the ellipse. If the lens has been fixed correctly to the axis D in this manner, it is then only necessary to rotate the couple by 90°, in order to obtain the middle position necessary for grinding where the axes assume the same position as in Fig. 2, when the latter is turned 180° ["stuerzen"].

If we want also to grind parallel surfaces of the second order, the axis B must then be capable of being displaced in the vertical direction. The vertical distance of the latter from the grinding surface is equal to the distance of the ground parallel surface from the surface of the second order.

The Duplex Method. Since only convex surfaces can be ground in this manner, this raises initially the question whether concave surfaces in a shape suitable for theoretical calculation can be produced with simple mechanical means. A first answer leads us to the composite curve and/or

circular conchoid generated by a thrust-crank drive and/or sliding-crank drive. The former is described in Fig. 4 by any desired point of the axis of revolution of the lens and the latter in Fig. 1 by any desired point of the line BC or its extension. In regard to simple construction, these curves therefore leave nothing to be desired and the conchoid has moreover already been utilized as machine curve in a patent[1]. Although both curves are only of the fourth order but offer even so no advantages in regard to calculation. In the Abbe curve with only one coefficient, the value of the latter results directly from the postulate for an osculation of the fourth order in the vertex and we should here desire that the respective value of the machine curve results from this postulate in an equally simple manner. The mathematical expression for such a condition becomes apparent in the following manner.

When ρ represents in general the radius of curvature and σ the length of arc of a plane curve, I have designated[2] the value

$$\Phi = \frac{d^2}{d\sigma^2}\frac{1}{\rho}$$

as the flattening value of the curve at the respective point. In the vertical point, the latter is dependent on the differential quotients of the second and fourth order and the postulate of a central osculation of the fourth order with a given curve is identical to the problem of determining the machine curve at a prescribed radius of curvature and flattening value in the vertex. If, in order to facilitate this operation as much as possible, the respective coefficient in the equation of the machine curve is to

1. D. R. P. N:o 214107 of G. OSSART and A. VERGE. Deutsche Mechaniker Zeitung, 1910, P. 91.
2. Allgemeine Theorie der monochromatischen Aberrationen und ihre nächsten Ergebnisse fuer die Ophthalmologie. Nova Acta Reg. Soc. Sc. Ups., Vol. 20, 1900.

be directly proportional to the prescribed flattening value, then this equation must be in such a form that the one coordinate is represented as a function of the other and that the differential quotient of the second order must vanish in the vertical point. In a finite curvature of the vertex, this is possible only with polar coordinates when the radius vector is represented as a function of the angle and the starting point is located in the center of curvature of the vertex. With an infinite radius of curvature of the vertex, the condition is fulfilled by the corresponding equation in Cartesian coordinates. A brief consideration shows that the respective machine curve cannot be generated by a simple plane mechanism. However, since these are the mechanisms making possible the simpliest calculations, the combination of such mechanisms appears to be the solution most advantageous from the viewpoint of theoretical calculation. In the polar equation $R = F(\beta)$ of the machine curve, for example, we can make the extension of the radius vector by a thrust-crank dependent upon an angle of rotation α in order to generate automatically the displacement through a thrust-crank corresponding to a trigonometric function of this angle from the eccentric angle β. The equation then assumes the form

$$\frac{R-R_0}{R_0} = C \cdot \varphi(\alpha) \qquad f(\alpha) = c \cdot \varphi(\beta)$$

in which R_o = radius of curvature of vertex and C, c = machine constants. The functions $\varphi(\alpha)$ and $\varphi(\beta)$ can be generated by thrust-cranks where the mathematical axes of the crank arm and of the couple in the position corresponding to the vertical point must coincide with the direction of the straight-line guidance. Both angles are calculated from this position and consequently represent the angles of rotation of the two crank-arms. The value of the radius vector of the machine curve is consequently dependent only on the absolute magnitude but not on the sign of the angle α and if we set,

for example f (α) as equal to sin α, then the value of α remains unchanged even with a change of sign of β. For $\beta = 0$, we then have $\frac{dR}{d\alpha} = \frac{d\alpha}{d\beta} = 0$, and, by differentiating four times, we obtain

$$\frac{d^2R}{d\beta^2} = 0 \qquad \frac{d^4R}{d\beta^4} = 3\frac{d^2R}{d\alpha^2}\left(\frac{d^2\alpha}{d\beta^2}\right)^2,$$

from which follows that the flattening value at the vertical point of the machine curve is directly proportionate to the constant C.

This advantage points out the necessity of examining more closely whether such curves are actually suitable for satisfying the demands made above on curves with several available constants. Since this is actually the case, I have selected curves of this type as machine curves under the general designation of duplex curves. Further investigation then showed that similar curves can be utilized advantageously in certain cases also if the differential quotient of the second order of the equation of the curve does not vanish in the vertical point. I therefore distinguish between two categories, depending on whether this is the case or not, and designate the curve represented by the equation above as a true duplex curve when the following conditions are complied with. The function $\varphi(\beta)$ must have a value of zero also for $\beta = 0$ but remains unchanged otherwise at a change of sign of β. Simultaneously with $f(\alpha)$, α must pass through the value of zero by having the differential quotient $f'(\alpha)$ different from 0. We merely require that the function $\varphi(\beta)$ vanishes together with the differential quotient of the first order at $\alpha = 0$. In the case of a non-focal non-spherical surface, R_o receives an infinitely large value so that the equation is written in the form

$$y = C \cdot \varphi(\alpha) \qquad f(\alpha) = c \cdot \varphi(x)$$

and the same requirements must be made on the different functions as when employing polar coordinates. In addition, I am introducing two categories

of pseudo-duplex curves which are represented by the equation

$$\frac{R-R_0}{R_0} = C_0 \cdot \varphi(\alpha) + C_1 \cdot \psi(\beta) \quad f(\alpha) = c \cdot \varphi(\beta)$$

and/or

$$y = C_0 \cdot \varphi(\alpha) + C_1 \cdot \psi(x) \quad f(\alpha) = c \cdot \varphi(x)$$

in which are valid the conditions indicated above for functions with the same sign ["gleichbezeichnet"] and the same requirements must be made on $\psi(\beta)$ and/or $\psi(x)$ as on $\varphi(\beta)$. In the application of certain mechanisms, pseudo-duplex curves can be represented also by equations of the same form as true duplex curves, except with this difference that the first differential quotient of the function $\varphi(\alpha)$ does not vanish at $\alpha = 0$.

A surface ground as machine curve by employing a duplex curve will be designated in general as a duplex surface. A true duplex surface is therefore characterized only by the fact that its meridian curve has a true duplex curve either among the parallel curves or as base curve in relation to the center of curvature of the vertex and the same is true of the pseudo-duplex surfaces represented by polar coordinates in relation to the pseudo-duplex curve, with this difference that a point on the axis other than the center of curvature of the vertex represents the point of the normal of the respective base curve. A pseudo-duplex curve represented by Cartesian coordinates is, if the surface is ground with one plane, actually only a curve derived from the respective base curve.

The duplex machine utilized for the grinding of the surfaces must contain in any case two different mechanisms, as will be seen from the foregoing, which we shall call the A- and B-mechanisms. Since the task of the A-mechanism consists in converting the rotation equal to α around the A-axis into a straight-line displacement of the machine component carrying either the abrading surface or the axis of rotation of the lens, it must therefore include straight-line guidance for which a carriage is preferable

for reasons already discussed. Depending on whether polar or Cartesian coordinates are utilized in the equation of the machine curve, the radius vector and/or the Y-axis must indicate the direction of the straight-line guidance. Should we desire for technical reasons to establish the path of the carriage as fixed, the lens must then be pivoted around the B-axis intersecting the axis of rotation of the lens perpendicularly in the center of curvature of the vertex, when utilizing polar coordinates, whereas the distance of this axis from the abrading surface is changed by the straight-line guidance. Without anticipating the question of whether greater advantage lies in establishing the B-axis or the abrading surface as fixed, let us assume initially, for the sake of easier comprehension of this representation, that the B-axis rotates in fixed bearings. Consequently, when grinding non-focal surfaces, the X-axis corresponding to the straight-line path along which the lens is guided, shall be considered as fixed. The abrading surface therefore executes a straight-line movement in these cases but another arrangement is preferable for grinding pseudo-duplex surfaces whose machine curves are given in Cartesian coordinates. If the function $\psi(x)$ is given a form such that the machine curve is transformed into a circle at $C_o = 0$, this function is directly generated most simply by rotating one machine component around a B-axis where the form $y = F(\beta)$ can be given to the equation of the machine curve. At a fixed path of the carriage, we can obtain fixed bearings for the B-axis only by pivoting the abrading surface around the B-axis whereas the lens displaces itself in the direction of its axis of rotation. In general, we shall therefore temporarily assume that the paths of the carriage of the A-mechanism and the B-axis are established as fixed. The B-mechanism in which we may understand as included also the function $f(\alpha)$ more properly connecting the

two mechanisms, is intended to automatically generate the rotation around the A-axis from that around the B-axis by complying with postulated mathematical conditions. As will be shown immediately, straight-line guidance is not absolutely necessary for this. However, such guidance offers certain advantages on the other hand so that it would seem indicated for certain purposes to accept the inclusion of a second carriage. On the basis of the assumption just formulated, the latter is always given a fixed path. In the interest of easier comprehension, let us assume further that, in the middle position, the axis of rotation of the lens is vertical and the abrading surface is located above the lens surface so that the direction of thrust of the two straight-line guidances becomes vertical.

In the examination of the different machine types applicable under the indicated conditions, let us start with the B-mechanism.

The obvious solution for generating the function $\varphi(\beta)$ is represented by the thrust-crank mechanism, possibly in a specialized form. Let us designate the length of the crank arm as A and compute it as positive if the axis of articulation of the latter is located below the B-axis as shown in Fig. 4. Let a + b be the length of the couple and computed as positive if the axis of articulation linked to the straight-line guidance is located above the other as in the figure. It follows from this that, in the initial position, b represents the distance from the B-axis of the axis of articulation linked to the straight-line guidance and must be computed as positive if the former is located above the latter. If the crank arm is rotated around the crank axis until it forms an angle β with the direction of the straight-line guidance, then γ will represent the angle now included by the couple with this direction. For the determination of the displacement in height l, we then obtain the two equations

$$l + b = (a + b)\cos\gamma - a\cos\beta \quad (a + b)\sin\gamma = a\sin\beta,$$

which are to be employed in the form

$$l = a\left(1 - \cos\beta - \frac{1}{k}(1 - \cos\gamma)\right) \qquad \sin\gamma = k\sin\beta$$

by setting $k = \frac{a}{a + b}$.

For the special cases corresponding to an infinite length of the couple and defined by the conditions k = 0 and/or k = 1, it is easy to carry out the following trigonometric conversion which is moreover advantageous for numerical calculation also in the general case. By taking into account that

$$1 - \cos\gamma = \sin\gamma\,\mathrm{tg}\,{}^1/_2\gamma \qquad \mathrm{tg}\,{}^1/_2\beta - \mathrm{tg}\,{}^1/_2\gamma = \frac{\sin{}^1/_2(\beta - \gamma)}{\cos{}^1/_2\beta\cos{}^1/_2\gamma}$$

we obtain through elimination of k:

$$l = \frac{2a\sin{}^1/_2\beta\sin{}^1/_2(\beta - \gamma)}{\cos{}^1/_2\gamma},$$

and this expression assumes, in the case of k = 0, the form

$$l = a(1 - \cos\beta)$$

On the other hand, through the elimination of k and a, we obtain with the aid of the relation

$$\sin\beta - \sin\gamma = 2\sin{}^1/_2(\beta - \gamma)\cos{}^1/_2(\beta + \gamma)$$

the expression

$$l = \frac{2b\sin{}^1/_2\beta\sin{}^1/_2\gamma}{\cos{}^1/_2(\beta + \gamma)},$$

to which is given, in the case of k = 1, the form

$$l = \frac{b(1 - \cos\beta)}{\cos\beta}$$

Further mention need only be made of the case k = -1 corresponding to a full rotation ["durchschlagend"] of the thrust-crank drive in which

$$l = 2a(1 - \cos\beta)$$

The thrust-crank drive which becomes transformed into a sliding-crank drive at k = 0 as well as at k = +/-1, can generally be replaced by utilizing a closed linkage through a curved-line guidance for which cylindrical surfaces are fixed on the carriage and on the crank arm which have their axes in the respective axes of articulation and are forced against each

other. The axis of the cylindrical surface fixed to the carriage therefore lies in the initial position at the distance b from the B-axis and this distance is considered as positive when the former is located above the latter. The axis of the cylindrical surface rigidly linked to the B-axis in turn lies at the distance A from the latter and this distance is considered as positive if the cylinder axis is located below the B-axis. The radius of one of the cylindrical surfaces can be selected arbitrarily so that three qualitatively different arrangements are possible in the general case, depending on whether the line of contact lies between the two cylinder axes or beyond one or the other. Of the three arrangements, at least one always permits utilization of gravity so that the carriage is carried by the B-axis at the line of contact of the cylinder surfaces. Only when the couple is so short that a pin with a diameter equal to the length of the couple would not be able to support the carriage, would it be impossible for this mechanical reason, at a positive value of a + b, to utilize the weight of the carriage, but in such cases curved-line-guidance is excluded for other reasons.

When employing curved-line guidance, it is preferable to substitute a roller, able to rotate around its axis, for one of the cylinders, in order to eliminate sliding friction. If this is done for the cylinder belonging to the B-axis, then the roller must be able to rotate around an axis parallel to the B-axis which is mounted on a crank arm of variable length. The carriage then rests on this roller by means of the surface affixed to the carriage which represents, depending on whether $k \gtreqless 0$, a cylindrical surface with convex underface, a plane, or a cylinder surface with concave underface. It is here assumed that $a > 0$ so that the middle position represents the highest position of the carriage when $k > 1$. However, if we desire to have a stable middle position at $k > 1$, we need only make a

negative but in that case the surface, by means of which the carriage rests on the roller, represents a cylinder surface with concave underface. The question is then whether this form of curved-line guidance is actually an advantage when the absolute value of k appreciably exceeds unity. By reason of the fact that here the angle of inclination of the couple is greater in absolute value than that of the crank, it will be better to have the force, inducing pivoting, attack at the couple which requires the retention of the thrust-crank drive without any change.

If a very high value of a is prescribed, the roller must be mounted on the carriage at the respective height. The surface linked to the B-axis on which it rolls, then represents at $b > 0$, a cylinder surface concave at the topface, a plane or a cylinder surface convex at the topface depending on whether $k \gtrless 1$. In those cases, curved-line guidance must therefore be employed also at $k > 1$ but in these cases the value of k differs very little from unity.

Obviously, it is also possible to employ another curved-line guidance in place of that corresponding to the crank drive where one of the two straight circular cylinders is replaced by a straight cylinder whose fundamental curve has a symmetrical axis. If this were done for both cylinders, calculations would become entirely too complicated and rolling would moreover not produce adequately positive guidance of motion so that sliding friction could not be prevented. The expression for the displacement in height of the carriage differs for the utilization of such cylinders depending on whether the roller is affixed to the carriage or to the crank arm. If the former is the case, the cylinder must be affixed to the crank arm so that its axis of symmetry contains the B-axis. If the crank arm is then rotated around this axis, then the axis of the roller affixed to the

carriage describes a curve in a plane rigidly linked to the crank arm and perpendicular to the B-axis which represents the parallel curve of the fundamental curve of the cylinder located at the distance of the radius of the roller. If this parallel curve is consequently given by an equation $r = f(\beta)$ where the respective point of the B-axis represents the pole and when r_o represents the value of r at $\beta = 0$, we then obtain for the displacement in height

$$l = f(\beta) - r_o$$

and must make $b = r_o$ in the machine.

On the other hand, if the roller is affixed to the crank arm and the cylinder to the carriage, then the plane of symmetry of the latter must be parallel to the direction of thrust and contain the B-axis. If the crank is rotated, then the axis of the roller affixed to the crank arm also describes a parallel curve of the fundamental curve of the cylinder in a plane perpendicular to the B-axis and rigidly linked to the carriage. The equation of this parallel curve shall be given in Cartesian coordinates where the X-axis is located in the axis of symmetry of the cylinder and is considered as positive upward whereas the Y-axis represents the tangent of the vertical point and consequently passes through the axis of the roller in the initial position. Customarily α is the length of the crank arm and the vertical point of the parallel curve in the initial position is then located at the distance α from the B-axis below the latter. After rotating the crank by an amount β, this distance is $\alpha \cos \beta + x$ and we thus obtain the displacement in height from the equations

$$l = a(1 - \cos\beta) - x \qquad y = a \sin\beta \qquad f(x,y) = 0.$$

On the basis of the conditions postulated, only such cylinders may be employed for this curved-line guidance which can be produced exclusively

by machining. It should be here noted in general that a kinematically producible curve may be employed as guiding curve if, in the machine generating the curve, the axis of the abrading cylinder describes the curve whereas the roller is given the same diameter as the abrading cylinder when employing a cylinder ground in this manner in the B-mechanism. The base curve of the cylinder so produced and so employed then is a parallel curve located at the same distance both from the guiding curve as well as the kinematically produced curve and these two curves are consequently identical. In this manner, for example, cylinders can be ground in an eccentric grinding device ["Ovalwerk"] which produce elliptic guiding curves in the B-mechanism and whose long or short axes can correspond to the initial position as desired. Moreover, since the method described above for the production of surfaces of the second order is also imminently suited to the grinding of cylindrical surfaces whose fundamental curves are parallel curves of the curves of the second order, conical sections can therefore be used quite generally as guiding curves in the B-mechanism.

It follows from the foregoing that the equation must be given in polar coordinates if the guiding curve is to be linked to the crank. The form of this equation most suitable for the present purpose is obtained from the familiar equation in Cartesian coordinates

$$y^2 = 2p(x + r_o) + q(x + r_o)^2$$

by substituting

$$x = -r\cos\beta \qquad y = r\sin\beta.$$

The value r_o assumed by the radius vector r at $\beta = 0$ therefore represents the distance of a vertical point from the pole of the coordinate system and ρ, the radius of curvature at this point, has a positive value if the center of curvature is located on the same side of the respective

vertical point as the pole. If $q > -1$, the equation $q = e^2 - 1$ then indicates the numerical eccentricity e. At a negative value of q, we have an ellipse whose semi-axis, coinciding and/or perpendicular to the initial line of the coordinate system, may be designated as A and/or B. In that case, $q = -\frac{B^2}{A^2}$ so that the shorter semi-axis coincides with the initial line of the polar coordinate system at $q < -1$. In solving the equation in regard to r, the sign of the square root must be selected so that $r = r_o$ is obtained also at $\beta = 0$. In this manner, we obtain for the displacement of the carriage $l = r - r_o$

$$l = \frac{r_0 (q \cos\beta - q\cos^2\beta + \sin^2\beta) + \rho(\cos\beta - u)}{q\cos^2\beta - \sin^2\beta},$$

in which u represents the positive root of the equation

$$u^2 = \cos^2\beta + \frac{r_0 \sin^2\beta}{\rho}\left(2 + \frac{qr_0}{\rho}\right)$$

If we here substitute e^2 for u and simultaneously set $\rho - r_o = \alpha$ which produces $k = \frac{\alpha}{\rho}$, then these equations can be expressed in the form

$$l = \frac{(a - r_0 e^2\cos\beta)(1 - \cos\beta) - \frac{a}{k}(1 - u)}{1 - e^2\cos^2\beta}$$

$$u^2 = 1 - \sin^2\beta\{k^2 - e^2(1-k)^2\}$$

from which it becomes directly apparent that the expression valid for the crank mechanism is obtained at $e = 0$ and can also be used at $q < -1$, provided we substitute $q + 1$ for e^2.

If a focus of the conical section represents the pole of the coordinate systems, we then have $r_o = \frac{\rho}{1 \pm e}$ from which results $\frac{k}{1-k} = \pm e$ and $b = \pm\frac{\alpha}{e}$. In that case, $u = 1$ and we obtain

$$l = \frac{a(1 - \cos\beta)}{1 \pm e\cos\beta},$$

where we must utilize the upper or the lower sign, depending on whether the pole coincides with the focus closer or more distant in relation to the vertical point.

Of other special cases, we need only note that we have also $l = 0$ at $r_o = 0$ as well as that the case corresponding to the condition $\Phi = 0$ for the machine curve, is characterized by $\alpha = k = 0$ and consequently $r_o = \frac{\alpha}{k} = \rho$ which gives us $u^2 = 1 + e^2 \sin^2 \beta$ so that the expression for l becomes simpler. The denominator becomes equal to zero then when β indicates the direction of asymptote of a hyperbola or is equal to zero in the parabolic equation. In the latter case, the value of l has the form $\frac{0}{0}$ and the corresponding differentiation produces the value 0.

Inversely, if the guiding curve is affixed to the carriage, then its equation in the indicated coordinate system is

$$y^2 = 2\rho x + q x^2$$

and we obtain

$$x = -\frac{\rho}{q}\left(1 - \sqrt{1 + q \sin^2 \beta \frac{a^2}{\rho^2}}\right),$$

in which ρ is also positive when the center of curvature is located above the vertical point. If we set $k = \frac{\alpha}{\rho}$, this produces

$$l = a\left(1 - \cos\beta + \frac{1}{kq}(1 - u)\right),$$

in which u represents the positive root of the equation

$$u^2 = 1 + k^2 q \sin^2 \beta$$

At $q = -1$, there results the expression valid for the crank mechanism and we obtain for the parabola

$$l = a\left(1 - \cos\beta - \frac{k \sin^2 \beta}{2}\right).$$

Except for curves of the second order, presently known curves can hardly be used to advantage in this manner because the calculations become too complicated. If this is not to be the case, then the equation of the curve must be able to assume either the form $r = f(\beta)$ as pole for any desired axial point or the form $x = f(y)$ as X-axis for the axis of symmetry. However,

these conditions are satisfied by the pseudo-duplex curves represented in polar and/or Cartesian coordinates so that they are highly suitable as guiding curves and consequently the respective cylinder can be ground in the machine itself by employing abrading cylinder of the same diameter as that of the roller.

If the diameter of the roller is changed when utilizing a guiding curve, then the fundamental curve of the cylinder represents the parallel curve located at the correspondingly changed distance. If the diameter of the roller increases appreciably and if sliding friction is permitted, then the roller can be replaced by a cylindrical segment affixed to the respective machine component. As long as the radius of the cylinder surface is finite, the formulas deduced above are valid without change. However, this is no longer the case, if the latter surface becomes transformed into a plane so that the guiding curve is infinitely distant. Consequently, a plane displaceable in the direction of its normal is held in this case in contact with a straight cylinder which is able to rotate around an axis perpendicular to the fundamental plane and, if the fundamental curve of the cylinder is a circle, the mechanism represents an eccentric and is therefore designated, for any desired form of the fundamental curve, as <u>eccentric mechanism</u>. If $r = f(\beta)$ represents the equation of the base curve of the fundamental cylinder in relation to the respective points of the axis of rotation, then the radius of vector is equal to the distance of the axis from the plane, from which it follows that the displacement of the plane corresponding to an angle of rotation β is equal to $r - r_o$ in which r_o customarily indicates the value of r at $\beta = 0$.

In the present demonstration, the most advantageous general form of the equation of the base curve results in the following manner. In a rectangular, plane coordinate system whose X- and/or Y-axis coincide with a normal of the

curve and/or with the tangent in the respective point of the curve, the magnitudes φ N M are to be defined by the equations

$$\cot\varphi = \frac{dy}{dx} \qquad N = \frac{y}{\sin\varphi} \qquad M = x + N\cos\varphi$$

in which that value of φ corresponding to the cotangent is to be selected which turns into zero when the point of the curve is guided along the curve to the initial point. N consequently represents the length of the normal and M the sum of subnormal and abscissa whereas the angle φ is formed by the normal with the X-axis. If this axis is a perfect normal, e.g., the axis of symmetry of the meridian curve of a surface of revolution, these three magnitudes then have the properties of intrinsic coordinates.

In order to find the equation $r = f(\beta)$ of the base curve in relation to a point on the X-axis, we draw both the tangent as well as the normal through any desired point of the curve and plot the vertical to both from the given axial point whose abscissa is equal to r_o. Projection to the normal produces

$$N = r + (M - r_0)\cos\varphi$$

and consequently, for the displacement l of the plane in the eccentric mechanism,

$$l = N - M\cos\varphi - r_0(1 - \cos\varphi),$$

in which φ represents the eccentric angle β. Since the base curve of the parallel curve, in relation to the point vertically below the point of the normal, represents in general a conchoid with the base curve of the original curve as base, l remains unchanged if a parallel curve of the given curve is employed instead of the latter. If we permit this curve to pass through the point of the normal, then r_o N M assume the values zero and/or $N_o = N - r_o$ and $M_o = M - r_o$ so that we obtain

$$l = N_0 - M_0\cos\varphi$$

These values of the displacement of the plane remain valid without change for the displacement of the carriage if the plane is affixed to the latter and the eccentric is affixed to the crank in which r_o consequently has a positive value if the line of contact is located above the B-axis. On the other hand, if the plane is affixed to the crank and the eccentric is affixed to the carriage, we then must divide the particular value of l by cos β, in order to obtain the displacement of the carriage from the formulas, and r_o must be considered as positive in the latter when the line of contact is located below the B-axis. If we should assume that the carriage is fixed and that the B-axis can be displaced in a vertical direction, there would then take place, with a rotation of the plane around this axis, a displacement of the latter in the direction of its normal in the amount of which would correspond to a vertical displacement of the B-axis in the amount $\frac{l}{\cos \beta}$.

For the same reasons as for curved-line guidance, only curves of the second order and duplex curves will be available for the eccentric mechanism. For the first of these curves, we obtain, by differentiation of the equation

$$y^2 = 2\rho x + q x^2$$

the value of the subnormal $\rho + qx$ whose square is $\rho^2 + qy^2$. Consequently, we have

$$N^2 = \rho^2 + e^2 y^2 \qquad M = \rho + e^2 x,$$

and elimination of x and y results in

$$N = \frac{\rho}{\sqrt{1 - e^2 \sin^2 \varphi}} \qquad M = \frac{e^2 N \cos \varphi - \rho}{q},$$

where e^2 at $q < -1$, is as everywhere in this demonstration, only an abbreviated designation for $q + 1$. The result is

$$N - M \cos \varphi = \frac{\rho}{q} (\cos \varphi - \sqrt{1 - e^2 \sin^2 \varphi}),$$

and the value

$$l = -\left(r_0 + \frac{\rho}{q}\right)(1 - \cos\beta) + \frac{\rho}{q}(1 - u),$$

in which u represents the positive root of the equation

$$u^2 = 1 - e^2 \sin^2\beta$$

i.e., the value $\frac{l}{\cos\beta}$ consequently furnishes the displacement of the carriage, depending on whether the eccentric is affixed to the crank or to the carriage.

At q = 0, the expression for l contains a number of the form $\frac{0}{0}$. However, from the above calculation, we obtain directly

$$N = \frac{\rho}{\cos\varphi} \qquad M = \rho\left(1 + \frac{\operatorname{tg}^2\varphi}{2}\right) \qquad N - M\cos\varphi = \frac{\rho \sin\varphi \operatorname{tg}\varphi}{2}$$

and can therefore write the expression for the parabolic eccentric in the form

$$l = \frac{\rho}{2}\left(\frac{1}{\cos\beta} - \cos\beta\right) - r_0(1 - \cos\beta)$$

The general expression which turns into the expression valid for a cylinder with a circular base at q = - 1 and assumes a particularly simple form when the B-axis passes through the center of the conical section which corresponds to the condition $r_o + \frac{\rho}{q} = 0$, can be written, as $q > -1$, in the form

$$l = a_1\left(1 - \cos\beta - \frac{1}{k_1}(1 - \cos\gamma)\right) \qquad \sin\gamma = \pm e \sin\beta$$

in which

$$a_1 = -\frac{\rho + q r_0}{q} \qquad k_1 = \frac{\rho + q r_0}{\rho}$$

and therefore coincides, at $k_1 = \pm e$, i.e., $r_o(1 \pm e) = \rho$ which indicates that the B-axis passes through a focus, with the expression valid for the crank mechanism. In fact, it is possible to grind with the crank mechanism

whose basis represents a conical section, when the abrading plane is affixed to the carriage and the ground cylinder to the crank. It thus becomes clear that the movement of the carriage must be precisely that generated by the crank mechanism when the cylinder acts as eccentric on the plane.

The eccentric mechanism has the advantage that it may eliminate a carriage in the B-mechanism under certain conditions. If we arrange the horizontal A-axis running in fixed bearings perpendicular to the B-axis, then a cylinder surface rigidly linked to the latter with its axis parallel to it, can rest directly on an eccentric rigidly linked to the B-axis so that the plane affixed to the carriage is replaced by the always horizontal tangential plane common to the crossed cylinders. This arrangement consequently requires the A-axis to be perpendicular to the B-axis but still retains the disadvantage that the form of the function $f(\alpha)$ cannot be as freely selected as when straight-line guidance is given.

As far as the function $f(\alpha)$ is concerned in general, the latter must represent above all such a specialization of the simple plane mechanism that calculation becomes as simple as possible and positive guidance as accurate as possible. From the former viewpoint, a more complicated function is to be preferred only then when real advantages are gained by the introduction of a new machine constant. This is the case in a generalization of the two most simple functions, i.e., sin α and tg α, but elsewhere the introduction of a new constant results only in very complicated calculations. I shall therefore discuss here only the two main types of the B-mechanism derived in this manner.

One of these, the general sinus mechanism is characterized by the function

$$f(\alpha) = \sin\alpha - \operatorname{tg}\omega(1 - \cos\alpha)$$

which is generated in the following manner. On a horizontal plane rigidly linked to the B-carriage, there rests a cylinder rigidly linked to the A-axis with its axis parallel to this axis and which can therefore be replaced by a roller, in order to prevent sliding friction if the possibility exists of mounting the supporting plane at any desired height on the carriage. The distance of the A-axis from the vertical plane in which the cylinder axis and/or the roller is located in the initial position, is designated with E_o whereas ω represents the angle formed with the horizontal by the plane drawn through the A-axis and the cylinder axis in the initial position and which is to be considered as positive upward. If the carriage is displaced upward by the distance l when the A-axis rotates around the angle α, then this plane assumes a position in which it forms the angle $\alpha + \omega$ with the horizontal. The height of the cylinder axis above the horizontal plane passing through the A-axis is the product of the distance of the two axes from each other and the sinus of the respective angles. Since, according to the definition, this distance is $\frac{E_o}{\cos \omega}$, the vertical displacement of the carriage is indicated by the equation

$$l = \frac{E_o}{\cos \omega} (\sin (\alpha + \omega) - \sin \omega)$$

which furnishes the expression above when $f(\alpha) = \frac{l}{E_o}$.

In the <u>general tangential mechanism</u> which is characterized by the function

$$f(\alpha) = tg (\alpha + \omega) - tg \omega$$

a cylinder is affixed to the carriage with the cylinder axis being parallel to the A-axis and on which rests a plane rigidly linked to the axis and parallel to the plane passing through the two axes. The plane parallel to the latter forms the angle ω with the horizontal in the initial position whereas E_o represents the distance of the A-axis from the plane in which

the axis of the cylinder moves.

In the general sinus mechanism, crossed cylinders may be utilized. It is here merely necessary for the A-axis perpendicular to the B-axis to be at a height sufficient for allowing absolute large negative values of ω. This angle is varied by changing the distance of the axes of the crossed cylinders from each other. If both cylinders have circular fundamental curves, this can be accomplished by changing the diameter either of only one or of both cylinders. However, if the eccentric represents a special cylinder, the vertex of the latter above the B-axis can -- as will be explained in detail further below -- be selected arbitrarily so that even in this case ω can be varied by changing either one or both of the cylinders.

However, even with the general tangential mechanism, the B-carriage can be eliminated under certain conditions if we utilize a toric eccentric. If the toric surface is regarded as the enclosing surface of a sphere whose center moves on the fundamental circle, then the distance of the A-axis from the plane perpendicular to the latter and rigidly linked to it which lies on the eccentric, must be equal to the radius of the sphere. The plane passing through the A-axis and the horizontal tangent of the fundamental circle then forms the angle $\alpha + \omega$ with the horizontal plane. In the expression for the upward displacement

$$l = a(1 - \cos \beta)$$

α is then the distance of the center of the torus from the B-axis and the angle ω results out of the diameter of the fundamental circle. However, this method has various disadvantages as compared to the crossed cylinders. In the first instance, the utilization of special cylinders is excluded and, in the second instance, the production is more difficult and, thirdly, mounting on the B-axis is more complicated because it is necessary to be

able to determine the distance E_o of the A-axis from the plane of the fundamental circle with maximum possible accuracy.

Since it appears from the foregoing that the function $\varphi(\beta)$ -- as far as it has been the subject of investigation -- can be represented by an expression of the form $\frac{\ell}{\alpha}$ in which α has the dimension of length whereas the function itself, except for the respective trigonometrical functions of β, contains only the coefficients k and q, so that the constant occurring in the equation $f(\alpha) = c \cdot \varphi(\beta)$ representing the B-mechanism results through elimination with the equation $f(\alpha) = \frac{\ell}{E_o}$ which furnishes $c = \frac{\alpha}{E_o}$. If we utilize a duplex curve as guiding curve or as fundamental curve of an eccentric (in which the cylinder must be ground with a plane in the latter case), the function $\varphi(\beta)$ and the constant c is to be formed in a similar manner.

In a B-mechanism as described above, we therefore have available at least one and at the most four machine constants -- disregarding the utilization of duplex curves which permit a greater number. The effective variability of the mechanism corresponding to these constants depends both on the mathematical means making possible the determination of the constants in a given case -- which will be discussed further below -- and also on the possibility of adapting one and the same machine to the different functions. In this respect, the carriage offers a great advantage because it allows not only the utilization of the crank but also makes possible the setting of the various angles ω in a mechanically more advantageous manner. For example, if the crank as shown in the diagram of Fig. 4 is arranged between the end of the B-axis and the carriage, the A-axis can be located parallel to the latter sufficiently high to permit any desired variation of ω in the sinus and/or the tangential mechanism. Within the limits controlling the construction or design of the machine, it is therefore possible, if the B-mechanism has a carriage, to have available three constants c ω k without

utilizing special cylinders nor does the utilization of any desired special cylinder encounter any difficulty so that, at the cost of such a cylinder, we would also have available the constant e and/or the constants of the duplex curve. Without a carriage, we are restricted to toric eccentrics and/or crossed cylinders but here the limits of variation of the angle ω are restricted for mechanical reasons since it is difficult to provide for a distance E_o sufficiently small to permit large absolute values of ω and/or $\alpha + \omega$. Because, when utilizing crossed cylinders, the appropriate width of the eccentric is equal to the difference of the maximum and the minimum value of $\left|\frac{E_o \cos(\alpha+\omega)}{\cos \omega}\right|$, the eccentric must be wider than when utilizing a carriage and this is a decisive factor particularly for special eccentrics. On the assumption that the absolute value of $\alpha + \omega$ required is not too high, however, it is possible to have available the constants c and ω even without a carriage and the utilization of a special cylinder together with a sinus mechanism then makes possible additionally the utilization of the constants k and e and/or of the constants of a duplex curve. The cylinder lying on the eccentric can also be replaced by a sphere whose center must then move in the plane containing the B-axis and perpendicular to the A-axis. In the initial position, the distance of the center of the sphere from the B-axis is equal to the magnitude b occurring in the equation of the crank mechanism whereas the distance of the former from the plane passing through the A-axis and parallel to the B-axis determines the angle ω. With fixed A-axis, the number of constants is consequently not increased and k varies instead through mathematical interrelations with ω. However, if the A-axis were capable of being displaced upward, then, even without a carriage, the three coefficients c ω k would be available within certain limits without the utilization of a special eccentric.

A merely theoretical interest is represented by the fact that we can here also replace the eccentric by a sphere which creates a simple spatial mechanism with closed linkage in which the spheres can be replaced by a rod which is linked, corresponding to the centers of the spheres, through universal joints with the crank arm and/or with a crank arm starting out from the A-axis.

If the equation of the machine curve is given in Cartesian coordinates, then the function $\varphi(x)$ occurs in the B-mechanism. If this then concerns an actual duplex curve, i.e., a non-focal surface, then the lens must be displaced back and forth on a horizontal path with unchanged direction of the axis of revolution. Although such displacement can be converted into the required vertical displacement of the B-carriage by a sliding cross-head drive, it is preferable to introduce a B-axis in order to exploit the existing possibilities and to generate the rotation around the latter in the most simple manner by displacement of the lens. This then furnishes the function $\varphi(x)$ from the function $\varphi(\beta)$ by means of one of the two equations $\sin \beta = \frac{x}{\alpha_o}$ and/or $\operatorname{tg} \beta = \frac{x}{\alpha_o}$ so that this case does not result in any change of the B-mechanism. The same is true for the pseudo-duplex curves in Cartesian coordinates where it is best (as already mentioned) to utilize a B-axis in the production and where the dependence of the angle β on the coordinate x is formulated in the function $\psi(x)$ so that $\varphi(x)$ is replaced simply by a function $\varphi(\beta)$. However, since the displacement effected by the A-mechanism is to act on the machine component carrying the axis of revolution of the lens, it is best to locate both the A-axis as well as the carriage of the B-mechanism below the B-axis since this causes only changes of detail in this mechanism.

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In the A-mechanism which is intended to convert the angle of rotation α into the prescribed straight-line movement, only the function $\varphi(\alpha)$ occurs

which must vanish together with the differential quotient of the first order at $\alpha = 0$. It will be evident that these conditions are satisfied by the different functions $\varphi(\beta)$ and that consequently the corresponding mechanism can also be applied to the A-mechanism. However, it will be equally evident especially here that great advantages for the determination of the machine constants could be procurred by an appropriate function. Actually, there exists the function $\varphi(\alpha)$, practically ideal from the mathematical viewpoint, which does, however, require the utilization of special cylinders in an eccentric mechanism. If the fundamental curve of the cylinder of the eccentric is a circular involute of any desired order, then we have available any desired number of machine constants and the latter are determined by a system of linear equations for various given problems. On the other hand, such cylinders can be produced by purely mechanical means where mechanical difficulties occur only when very small radii of evolutes and/or cusps are involved. The abrading tool ["fraize"] and/or the abrading roller can run in fixed bearings but the "band" must start from a fixed point located on the extension of the axis and the cylinder to be ground must be coupled rigidly to the evolute cylinder utilized. Both cylinders are best mounted on a common axis whose bearings are supported by a carriage which can be displaced in a direction along this axis and perpendicular to the axis of the abrading surface. The development can then take place on the one and the grinding on the other side of the carriage. In order to be able to generate involute radii as small as possible, the plane containing the axis should be parallel to the path of the carriage. In dimensioning the length of the band, it should be remembered that the curve described by one point of the axis of the abrading surface represents the parallel curve, located at the distance of the radius of this surface, of the ground curve. The accuracy with which

the length of the band can be adjusted can be determined only by practical experience. As far as the involutes of the first order are concerned, an exact length of the band is not necessary since these involutes are identical with their parallel curves so that an error in the length of the band is corrected by utilization of the correct point of the involute.

The equations defining M and N as indicated above, furnish in the form

$$N = \frac{y}{\sin \varphi} \qquad M = x + y \cot \varphi$$

by taking into account that if ρ represents the radius of curvature in general we have

$$dy = \rho \cos \varphi d\varphi \qquad dx = \rho \sin \varphi d\varphi$$

in accordance with the given definitions, the value

$$N - M \cos \varphi = \sin \varphi \int \rho \cos \varphi d\varphi - \cos \varphi \int \rho \sin \varphi d\varphi$$

for the displacement of the eccentric. Let

$$\rho = \sum_{n=0}^{n=m} a_n \frac{\varphi^n}{n!},$$

be the equation of a circular involute of the order m in which consequently α_o represents the radius of curvature of the curve at the point φ = 0 but that otherwise α_n represents the radius of curvature corresponding to the same point of the n-th successive evolute and in which the symbolic designation zero [!] is equal to unity. We therefore have

$$\int \rho \cos \varphi d\varphi = \sum_{n=0}^{n=m} \frac{a_n}{n!} \int \varphi^n \cos \varphi d\varphi$$

in addition to the analog expression for the other integral. The familiar reduction formula

$$\int \varphi^n \cos \varphi d\varphi = \varphi^n \sin \varphi + n\varphi^{n-1} \cos \varphi - n(n-1) \int \varphi^{n-2} \cos \varphi d\varphi$$

results in both for even-number n

$$\frac{a_n}{n!}\int \varphi^n \cos\varphi\, d\varphi = \pm a_n \sin\varphi\left(1-\frac{\varphi^2}{2!}+\frac{\varphi^4}{4!}-\cdots\pm\frac{\varphi^n}{n!}\right)\mp$$

$$\mp a_n \cos\varphi\left(\varphi-\frac{\varphi^3}{3!}+\frac{\varphi^5}{5!}-\cdots\mp\frac{\varphi^{n-1}}{(n-1)!}\right),$$

where the upper and/or lower signs are to be used, depending on whether n is or is not divisible by four without residue, and also for odd-number n

$$\frac{a_n}{n!}\int \varphi^n \cos\varphi\, d\varphi = \pm a_n \sin\varphi\left(\varphi-\frac{\varphi^3}{3!}+\frac{\varphi^5}{5!}-\cdots\pm\frac{\varphi^n}{n!}\right)\pm$$

$$\pm a_n \cos\varphi\left(1-\frac{\varphi^2}{2!}+\frac{\varphi^4}{4!}-\cdots\pm\frac{\varphi^{n-1}}{(n-1)!}\right),$$

in which the upper and/or lower signs are to be used, depending on whether n divided by four furnishes a residue of 1 or 3. If we now define, through the equations

$$c_n = a_n - a_{n+2} + a_{n+4} - \cdots$$

which all terminate in the member α_m and/or α_{m-1}, the m + 1 magnitudes c_n, the sum of the integral can then be given the form

$$\sum_{n=0}^{n=m}\frac{a_n}{n!}\int \varphi^n \cos\varphi\, d\varphi = \sin\varphi\sum_{n=0}^{n=m} c_n\frac{\varphi^n}{n!} + \cos\varphi\sum_{n=0}^{n=m-1} c_{n+1}\frac{\varphi^n}{n!} - c_1$$

in which the integration constant is determined by the fact that the sum must vanish at $\varphi = 0$.

In the same manner, the reduction formula

$$\int \varphi^n \sin\varphi\, d\varphi = -\varphi^n\cos\varphi + n\varphi^{n-1}\sin\varphi - n(n-1)\int \varphi^{n-2}\sin\varphi\, d\varphi$$

furnishes for even-number n

$$\frac{a_n}{n!}\int \varphi^n \sin\varphi\, d\varphi = \mp a_n \cos\varphi\left(1-\frac{\varphi^2}{2!}+\frac{\varphi^4}{4!}-\cdots\pm\frac{\varphi^n}{n!}\right)\mp$$

$$\mp a_n \sin\varphi\left(\varphi-\frac{\varphi^3}{3!}+\frac{\varphi^5}{5!}-\cdots\mp\frac{\varphi^{n-1}}{(n-1)!}\right)$$

and for odd-number n

$$\frac{a_n}{n!}\int \varphi^n \sin\varphi\, d\varphi = \mp a_n \cos\varphi\left(\varphi-\frac{\varphi^3}{3!}+\frac{\varphi^5}{5!}-\cdots\pm\frac{\varphi^n}{n!}\right)\pm$$

$$\pm a_n \sin\varphi\left(\varphi-\frac{\varphi^2}{2!}+\frac{\varphi^4}{4!}-\cdots\pm\frac{\varphi^{n-1}}{(n-1)!}\right),$$

where the signs in these equations must be applied according to the rules indicated above. Summation results in

$$\sum_{n=0}^{n=m} \frac{a_n}{n!} \int \varphi^n \sin \varphi \, d\varphi = -\cos \varphi \sum_{n=0}^{n=m} c_n \frac{\varphi^n}{n!} + \sin \varphi \sum_{n=0}^{n=m-1} c_{n+1} \frac{\varphi^n}{n!} + c_0,$$

where the constant of integration is also determined by the fact that the sum must vanish at $\varphi = 0$. We further obtain

$$\sin \varphi \int \rho \cos \varphi \, d\varphi - \cos \varphi \int \rho \sin \varphi \, d\varphi = \sum_{n=0}^{n=m} c_n \frac{\varphi^n}{n!} - c_0 \cos \varphi - c_1 \sin \varphi,$$

and this expression, if φ is replaced by α, indicates the amount of displacement in the straight-line guidance of the A-mechanism if a cylinder with the respective circular involute as fundamental curve is used as eccentric in the latter. As was demonstrated above, the magnitudes M N here have a fixed relation to the parallel curve passing through the A-axis of the fundamental curve of the cylinder so that α_o represents the distance of the center of curvature of the fundamental curve from the A-axis and is considered positive when this point is located below the axis. In concordance with this, the eccentric must be affixed to the axis so that the normal determined by the value α_o is intersected by the axis in the point determined precisely by this value and, at $\alpha = 0$, is perpendicular to the plane on which the eccentric acts. If we set the displacement as equal to $c_o \varphi(\alpha)$, we then obtain

$$\varphi(\alpha) = 1 - \cos \alpha + k_1(\alpha - \sin \alpha) + \sum_{n=2}^{n=m} k_n \frac{\alpha^n}{n!},$$

in which the numbers $k_n = \frac{c_n}{c_o}$ represent the available machine constants contained in this function.

The practical importance of this eccentric mechanism will be shown only by experience. As will be shown further below, machine constants are generally available, in a number adequate for most cases, without the

utilization of special cylinders and it therefore seems probable that circular involutes of a higher order than the first order will need to be employed only in relatively infrequent exceptional cases.

In those cases where machine constants are not required in the function $\varphi(\alpha)$, this function is most simply made equal to $1 - \cos \alpha$ and generated by a standard eccentric. It should be noted that, if α is obtained by the sinus mechanism, the bearings of the A-axis can then be mounted on the carriage so that the eccentric acts downward on a fixed plane. However, in the calculation of the angle ω, the axis of the cylinder of the eccentric then takes the place of the A-axis. The motion of the eccentric and the machine components coupled to the latter can be decomposed into rotation around the axis of the cylinder and a horizontal displacement of this axis. When utilizing the general sinus mechanism, however, horizontal displacement has no influence on the angle α because the respective machine component with another cylinder rests on an horizontal plane also. The process is therefore the same as if the axis of the eccentric were fixed. This arrangement can be utilized for the purpose of enlarging the numerical value of a negative angle ω at an equally high position of the A-axis.

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As a <u>compound duplex machine</u>, I designate a machine in which a straight-line motion is generated through summation of the effects of two or more individual mechanisms. In a general case, such a summation must take place in the production of the pseudo-duplex curves given in polar coordinates where the extension of the radius vector of the machine curve is equal to the sum of the displacement effected by the functions $\psi(\beta)$ and $\varphi(\alpha)$. If $\psi(\beta) = \varphi(\beta)$, it is only necessary to mount the bearings of the A-axis on the carriage of the B-mechanism. The rotation of the latter can then be effected

arbitrarily by the general sinus or tangential mechanism by forcing the cylinder and/or the respective plane rigidly coupled to the latter, in an upward direction against a fixed horizontal plane and/or fixed cylinder. If the A-mechanism consists of a standard eccentric, the bearings of the A-axis can be mounted, in accordance with the method just described, also on the carriage of the A-mechanism in which the eccentric acts downward on a horizontal plane coupled to the carriage of the B-mechanism, provided, however, that the sinus mechanism is used. If the crank mechanism is not utilized in the function $\varphi(\alpha)$, the A-carriage can be located above the B-carriage so that both carriages could run in one and the same path if this were a mechanical advantage. If both the A- and the B-mechanism consist of standard eccentrics and the sinus mechanism is employed, one of the carriages can even be eliminated by utilizing the method of the crossed cylinders. The A-axis must then be perpendicular to the B-axis and the A-eccentric directed downward then rests directly on the B-eccentric directed upward.

If we select, in order to have available one more machine constant, different functions for $\psi(\beta)$ and $\varphi(\beta)$, the arrangement just described must be modified so that the rotation of the A-axis is no longer effected by forcing a machine component coupled to this axis against a fixed machine component and the latter must be set in motion instead through a special B-mechanism. The displacement effected by this second B-mechanism may then take place either in horizontal or in vertical direction. In the first case, the sinus mechanism must be employed by mounting a vertical plane parallel to the B-axis on the carriage displaceable in a horizontal direction and on which acts the cylinder parallel to the B-axis which is coupled to the A-axis parallel with the same axis. The second B-mechanism in this

case directly produces the function $\varphi(\beta)$. However, if the straight-line guidance of this mechanism is vertical -- when we can also employ the tangential mechanism -- we obtain this function from the difference of the displacements effected through the two B-mechanisms.

In the case of $\varphi(\alpha) = 1 - \cos\alpha$ and $\varphi(\beta) = \psi(\beta)$ and where $f(\alpha)$ represents the sinus mechanism, the pseudo-duplex curve represented in polar coordinates can also be generated in a standard duplex machine. The equation of the latter can be written in this case in the form

$$\frac{R - R_0}{R_0} = C\{\cos\delta - \cos(\delta + \alpha)\} \quad \sin\alpha - \operatorname{tg}\omega(1 - \cos\alpha) = c\varphi(\beta)$$

which is demonstrated in the following manner. By elimination of $\sin\alpha$, we obtain

$$\cos(\delta + \alpha) = \cos\delta\cos\alpha - \sin\delta\{c\varphi(\beta) + \operatorname{tg}\omega(1 - \cos\alpha)\}$$

and therefore

$$\frac{R - R_0}{R_0} = C\{(\cos\delta + \sin\delta\operatorname{tg}\omega)(1 - \cos\alpha) + c\sin\delta\varphi(\beta)\},$$

from which results, by taking into account that

$$\cos\delta + \sin\delta\operatorname{tg}\omega = \frac{\cos(\delta - \omega)}{\cos\omega}$$

the equation

$$\frac{R - R_0}{R_0} = C_0(1 - \cos\alpha) + C_1\varphi(\beta)$$

in which

$$C_0 = C\frac{\cos(\delta - \omega)}{\cos\omega} \quad C_1 = cC\sin\delta$$

If the machine constants C_0, C_1, c, ω are given, we therefore obtain δ and C from the equations

$$\cot\delta = \frac{cC_0}{C_1} - \operatorname{tg}\omega \quad C = \frac{C_1}{c\sin\delta}.$$

The standard eccentric serves for generating the function cos δ - cos ($\delta + \alpha$) where we need only provide that, in the initial position at $\alpha = 0$, the plane containing the A-axis and the axis of the cylinder of the eccentric forms the angle δ with the vertical plane and that this angle is considered positive in the same direction as α. If α represents the distance of the two axes from each other, then $C = \frac{\alpha}{R_o}$.

This method makes it possible to grind non-focal surfaces with the standard duplex machine where the curvature of the vertex of the machine curve can be made equal to zero. Condition for this is that the value of $\frac{d^2R}{d\beta^2}$ for $\beta = 0$ becomes equal to R_o which is satisfied when $C_1\varphi_0''(\beta) = 1$.

The importance of the compound duplex machines lies in the fact that these can impart properties to the machine curve which can otherwise not be achieved, except by utilizing special cylinders. For example, if we require of a machine curve that the radius vector at a given finite inclination against the axis shall have the same value as on the axis, this would be possible in a standard duplex machine without special cylinder only by a full revolution of the A-axis. This could be achieved only by making the functions $f(\alpha)$ equal to α and by generating it through rolling off which would be, however, a disadvantage both in mechanical as well as in mathematical respect. However, if a straight-line motion in the machine is composed of two separate such movements, then the range of the latter also includes machine curves of this type.

If such compounding takes place in the A- and/or B-mechanism of a duplex machine, the latter will be designated as an A- and/or B-<u>triplex machine</u>. If D represents in general a function of the form

$$D = \psi(\varepsilon) + k\varphi(\gamma) \qquad f(\gamma) = c_1\varphi(\varepsilon)$$

in which are valid for the respective separate functions the same conditions

established above for functions of equal sign, a triplex machine is consequently defined by the fact that the function $\varphi(\alpha)$ and/or $\varphi(\beta)$ is equal to D in which α and/or β assumes the place of ε. The corresponding mechanical characteristic is the existence of a third axis, the G-axis, which is rotated by the amount of the angle γ. What has been said above on the compound duplex machine is valid also for the compound A- and/or B-mechanism. It follows from this that, when employing the sinus mechanism for $f(\gamma)$, the G-axis can run in fixed bearings if $\psi(\varepsilon) = \varphi(\varepsilon)$ and $\varphi(\gamma) = 1 - \cos\gamma$, but that, if these conditions are not satisfied, the bearings must be mounted on the carriage of the respective mechanism. To enter on the details of the arrangements mechanically possible would lead us too far afield here because actually many different types offer themselves. Let us stress merely that generally one of the three straight-line guidances can be replaced by the method of the crossed cylinders. An A-triplex machine can even be built with only one carriage. A simple but efficient B-triplex machine is obtained from a standard duplex machine with two carriages where the bearings of the G-axis are mounted at the top of the B-carriage and the cylinder rigidly coupled to the A-axis rests on an eccentric coupled to the G-axis. Although it is here necessary that $f(\alpha)$ represents the sinus mechanism, this is not a disadvantage as will be shown further below in the determination of the machine constants. The same is true of the condition generally inherent in this arrangement that $\alpha + \omega$ may not assume too great a numerical value.

Since the function D represents a pseudo-duplex curve, it is evident that a triplex machine can be replaced by a duplex machine if we employ in the latter a corresponding <u>duplex cylinder</u> for curved-line guidance. This designation is intended to signify in general a cylinder whose fundamental

curve represents the meridian curve of a duplex surface and which can therefore be ground in a duplex machine the fundamental curve of the cylinder must therefore be in this case the parallel curve of a pseudo-duplex curve; the duplex cylinder must be mounted, depending on whether this curve is given in polar or in Cartesian coordinates, on the respective axis and/or carriage; and the diameter of the cylinder rolling on the latter must be equal to that of the abrading cylinder which was utilized in the production of the duplex cylinder. That the duplex cylinder in operation must further assume exactly the same position in relation to the respective axis as in the production of the B-axis, will be automatically evident. The triplex machine, however, can also be replaced by a duplex machine if a corresponding duplex cylinder is utilized in the latter as eccentric. This eccentric must then have been ground with a plane and the fundamental curve may represent, if $\psi(\varepsilon)$ in the function D is made equal to 1 - cos ε, the base curve of a true duplex curve in relation to the center of curvature of the vertex. For example, if we desire to generate the machine curve of an A-triplex machine

$$\frac{R-R_0}{R_0} = C\{1-\cos\alpha + k_1\,\varphi(\gamma)\} \qquad f(\gamma) = c_1\,\varphi(\alpha) \qquad f(\alpha) = c\,\varphi(\beta)$$

in a duplex machine, we first grind a duplex cylinder with one plane by utilizing the machine curve

$$\frac{R-R_0}{R_0} = C\,\varphi(\alpha) \qquad f(\alpha) = c_1\,\varphi(\beta)$$

In this equation, the functions $\varphi(\alpha)$, $f(\alpha)$, $\varphi(\beta)$ must here be the same as $\varphi(\gamma)$, $f(\gamma)$, $\varphi(\alpha)$ in the former and c_1 must have one and the same value in both equations whereas it is sufficient otherwise that the product CR_0 in the second equation is equal to the product Ck_1R_0 in the first. The cylinder thus ground is then incorporated in the A-mechanism as eccentric by making the distance of the A-axis from the center of curvature at the vertex equal to the product CR_0 formed from the coefficient of the first equation.

That the action of the eccentric here corresponds to this equation will be seen from the equation given above on page 47 for the displacement ℓ. Since consequently the radius of curvature of the vertex of the duplex eccentric can be selected freely, we are able to influence the distance of the vertex from the A-axis. Whether we should prefer the employment of the duplex cylinder as eccentric or for the purpose of curved-line guidance will depend on given circumstances. The eccentric has the disadvantage of sliding friction which can be eliminated through curved-line guidance but operates more satisfactorily than a steeply rising guidance curve because of more favorable transmission of power.

It will be evident that duplex cylinders can be ground also with machines in which such cylinders are employed and this would correspond to the utilization of a compound machine with four axes and this procedure can be continued ad infinitum. As we intend to demonstrate further below, the same purpose can be accomplished in this manner as with involute eccentrics of higher order and we obtain with both methods the same number of machine constants available in the same manner if the same number of special cylinders is ground.

By this demonstration of the different possibilities of application of the duplex method, I intended to give the necessary review for being able to investigate the methods of determination of machine constants. A selection among the individual types or a detailed evaluation of the latter can be effected only from a knowledge of these methods. Here we desire to stress in this respect only that, the simpler the A-mechanism, the easier it will be to grind lenses of one and the same type with different curvature of the vertex. If the A-mechanism consists of a standard eccentric, we then need change only the latter (and the abrading surface correspondingly) to convert to another curvature of the vertex. From the same point

of view, grinding with one plane is preferable for convex surfaces whose meridian curves do not have points of inflection because we then need only change the eccentric for a change in curvature of the vertex.

In standard eccentrics, this change signifies only the adjustment of the prescribed distance of the cylinder axis from the machine axis. This change can be accomplished by a change in the length of a crank arm where the cylinder can rotate around its axis in order to eliminate sliding friction and may have any desired diameter.

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III - Determination of Machine Constants

Depending on the objective to be accomplished by the introduction of a non-spherical surface in an optical instrument, we will require different expressions for the demands made on this surface. If we are concerned merely with the correction of a Seidel image error, only the flattening value of the meridian curve in the vertex is prescribed. In other cases, e.g., surfaces of the second order or Cartesian ovals, the equation of the meridian curve is given in Cartesian coordinates. If we speak generally of an osculation of the order zero when the meridian curve of the ground surface intersects that of the prescribed surface in a given point, we can then express the problem as the intention of deriving a centric osculation of prescribed order and a number of eccentric osculations of also prescribed order. As will be explained in further detail below, it is theoretically possible to satisfy any desired one of these prescriptions by employing circular-involute eccentrics or duplex eccentrics of higher order. However, in practice, the objective is always achieved with a restricted number of machine constants. Even when the meridian curve of the non-spherical surface can be constructed only punctually, the problem is given the same expression. The direction of the normal and the radius of curvature generally results from the calculation for punctual construction but can, if this should not be the case, be determined with any desired accuracy by numerical methods and the same is true of the differential quotients of higher order in the vertex so that even in this case a central osculation of higher order can be prescribed. When employing a restricted number of machine constants, the problem must be given a special formulation. Depending on whether this formulation is restricted to the centric or an eccentric osculation, entirely different methods of calculation result so that it seems indicated to treat these problems separately.

Centric Osculation of Higher Order. In order to be able to employ the maximum possible number of machine constants for centric osculation, the simplest possible coordinate system must be selected because the complexity of calculation increases in proportion to the increase in order of osculation. For this reason, we prefer the polar coordinate system forming the basis of the typical duplex curve and whose pole is located in the center of curvature of the vertex. Because the machine curve whose equation has the form $R = f(\beta)$ with the condition $R'' = 0$ for $\beta = 0$, represents a parallel curve of the meridian curve of the ground surface and/or the base curve of the latter in relation to the center of curvature of the vertex, the differential quotients of higher order of this equation must be determined from the corresponding differential quotients of the equation of the meridian curve of the prescribed surface. If the equation of the meridian curve was given in the form $r = f(\theta)$ and if $r'' = 0$ at $\theta = 0$ in the latter which corresponds to the condition that the center of curvature of the vertex should represent the pole of the coordinate system, then the problem would be restricted to calculating the corresponding differential quotients of a parallel curve and/or the respective base curve from the differential quotients of this equation valid for the vertex. However, since this is not the case in general, we therefore must determine first an equation

$$r - r_0 = \sum \frac{d^n r}{d\theta^n} \cdot \frac{\theta^n}{n!} \qquad n = 4, 6, 8 \ldots$$

in which r_o represents the value assumed by r at $\theta = 0$ and which must be equal to the radius of curvature of the vertex. In this equation, the highest value of n indicates the prescribed order of osculation and the corresponding differential quotients which we will designate as $r^{IV} r^{VI} \ldots$ for the sake of brevity, must be calculated from the data prescribed for the non-spherical surface. If the latter is constructed punctually, there is

then known in any event the radius of curvature of the vertex and r_o is thus determined. The coordinates of $\frac{n}{2} - 1$ or, if the flattening value has been determined, those of $\frac{n}{2} - 2$ points are expressed in $r\theta$ from which we obtain a corresponding number of linear equations, by substitution of these values in the above equation, and the differential quotients result from these. Compensation calculation by utilizing the coordinates of a greater number of points would be of no advantage because the curve represented by the equation is not identical with the curve ground but has only a contact of the n-th order with the latter so that a correction, when required, is carried out only in the machine constants. We can proceed in the same manner when the equation of the meridian curve of the prescribed surface is given in such a form that the latter cannot be expressed by Cartesian coordinates. For example, this would be the case with a transcendant equation in polar coordinates if the second differential quotient in the vertex had a finite value. The direct calculation of the differential quotients r^{IV} ... from those of such an equation could be effected but would doubtlessly be without practical significance.

This calculation assumes the following form when the equation of the meridian curve is given in Cartesian coordinates. Let such an equation in the coordinates $\xi\eta$ be referenced to a coordinate system whose Y-axis coincides with the axis of symmetry of the curve and is considered positive in the direction from vertex to the center of curvature of the vertex whereas the X-axis represents the tangent of the vertex. The differential quotients determined from the latter shall be designated as $\eta' \eta'' \eta''' \eta^{IV}$ ... Through the substitutions

$$\eta = r_0 - r\cos\theta \qquad \xi = r\sin\theta,$$

in which r_o has the value of the radius of the curvature of the vertex, and

through subsequent successive differentiation, it is now possible to determine the differential quotients r^{IV} ... corresponding to the vertex but it is preferable to make this substitution calculation, so-to-speak, once and for all by deriving the formulas through which the differential quotients r^{IV} ... are obtained directly from the values of $\eta''\eta^{IV}$... in the vertex. For this purpose, the substitution equations are to be differentiated by considering one of the variables θ or ξ as independent. For greater ease in calculation, let θ be considered an independent variable. Since the Y-axis represents an axis of symmetry of the curve, all differential quotients $\frac{d^n\eta}{d\xi^n}$ in the vertex are equal to zero when n represents an odd-number figure, and the successive differentiation of the first equation shows that this is the case under the same condition also for the differential quotients $\frac{d^n r}{d\theta^n}$. In concordance with this, the differential quotients $\frac{d^n\xi}{d\theta^n}$ vanish for even-number n which becomes apparent also in the successive differentiations of the second order. The differential equations of odd-number order therefore vanish for the first equation of substitution and those of the even-number order for the second of these equations. The equations obtained during the first step for $\theta = 0$

$$d^2\eta = -d^2r - r_0 d^2\cos\theta \qquad d\xi = r_0 d\sin\theta$$

furnish

$$\eta'' r_0^2 = -r'' + r_0,$$

from which follows $r'' = 0$ because r_0 is equal to the radius of curvature and η'' is equal to the reciprocal value of the latter. From further differentiations of the second equation, we obtain, if we set $\theta = 0$ after the differentiation

$$d^3\xi = r_0 d^3\sin\theta \qquad d^5\xi = r_0 d^5\sin\theta + 5d^4 r\, d\sin\theta$$
$$d^7\xi = r_0 d^7\sin\theta + 35 d^4 r\, d^3\sin\theta + 7 d^6 r\, d\sin\theta$$

and therefore have

$$d\xi = r_0 d\theta \qquad d^3\xi = -r_0 d\theta^3$$
$$d^5\xi = (r_0 + 5r^{IV})\,d\theta^5 \qquad d^7\xi = (-r_0 - 35r^{IV} + 7r^{VI})\,d\theta^7.$$

In the same manner, further differentiation of the first equation for $\theta = 0$ produce

$$d^4\eta = -d^4r - r_0 d^4 \cos\theta$$
$$d^6\eta = -d^6r - 15d^4r\,d^2\cos\theta - r_0 d^6\cos\theta$$
$$d^8\eta = -d^8r - 28d^6r\,d^2\cos\theta - 70d^4r\,d^4\cos\theta - r_0 d^8\cos\theta,$$

in which

$$d^4\eta = \eta^{IV}d\xi^4 + 4\eta''d\xi d^3\xi$$
$$d^6\eta = \eta^{VI}d\xi^6 + 20\eta^{IV}d\xi^3 d^3\xi + 10\eta''(d^3\xi)^2 + 6\eta''d\xi d^5\xi$$
$$d^8\eta = \eta^{VIII}d\xi^8 + 56\eta^{VI}d\xi^5 d^3\xi + 280\eta^{IV}d\xi^2(d^3\xi)^2 + 56\eta^{IV}d\xi^3 d^5\xi + 56\eta''d^3\xi d^5\xi + 8\eta''d\xi d^7\xi$$

as well as

$$d^2\cos\theta = -d\theta^2 \qquad d^4\cos\theta = d\theta^4 \qquad d^6\cos\theta = -d\theta^6 \qquad d^8\cos\theta = d\theta^8$$

must be set. Since $d^4r = r^{IV}d\theta^4$, etc., we need only substitute the values of $d\xi d^3\xi\ldots$ in order to obtain the formulas. In the form most favorable for numerical application, the latter are

$$r^{IV} = -\eta^{IV}r_0^4 + 3r_0$$
$$r^{VI} + 20r^{IV} = -\eta^{VI}r_0^6 + 15\eta^{IV}r_0^4$$
$$r^{VIII} + 28r^{VI} + 329r^{IV} - 280\frac{r^{IV^2}}{r_0} = -\eta^{VIII}r_0^8 + 56\eta^{VI}r_0^6 - 315\eta^{IV}r_0^4.$$

Because the flattening value $\Phi = \eta^{IV} - 3\eta''^3$, we must make

$$r^{IV} = -r_0^4\Phi$$

if the former is prescribed. I have also determined the relations of the differential quotients to the flattening values of higher order $\frac{d^4}{d\sigma^4}\frac{1}{\rho}$ and $\frac{d^6}{d\sigma^6}\frac{1}{\rho}$ through differentiation of the general equation for the radius of curvature both in Cartesian and in polar coordinates. Comparison of the values so obtained shows that no error is contained in the above calculation.

In this way, we therefore always know the differential quotients of an equation $r = f(\theta)$ representing the meridian curve of the prescribed surface and in which $r'' = 0$ for the vertex. In order to obtain from them the corresponding differential quotients of the equation $R = f(\beta)$ of the machine curve, let us assume first that this curve represents a parallel curve of the letter located at the distance α from the meridian curve and where this distance is considered as positive when the radius of curvature of the machine curve is greater in the vertex than that of the meridian curve. If φ is the angle formed by a normal common to the two curves with the symmetry axis and if Rr are the radius vectors of the points of the curve determined by this normal, we then obtain through projection of the latter both to the normal and the tangent the two equations

$$R\cos(\beta-\varphi) = r\cos(\theta-\varphi) + a \qquad R\sin(\beta-\varphi) = r\sin(\theta-\varphi)$$

and obtain moreover the familiar relations

$$R'\cos(\beta-\varphi) = R\sin(\beta-\varphi) \qquad r'\cos(\theta-\varphi) = r\sin(\theta-\varphi).$$

These equations are now to be differentiated by treating β as an independent variable. The first differentiation of the second and of the fourth equation furnishes $d\theta = d\varphi = d\beta$ from which is obtained $R'' = 0$ by two differentiations of the first equation. The second equation is now no longer necessary. Because of the symmetry, the derivations of odd-number order of the first and those of the even-number order of the last two equations vanish. Successive differentiations of the last equation produce

$$d^3r' = r_0 d^3\sin(\theta-\varphi) = r_0(d^3\theta - d^3\varphi)$$
$$d^5r' = r_0 d^5\sin(\theta-\varphi) = r_0(d^5\theta - d^5\varphi),$$

where

$$d^3r' = r^{IV}d\theta^3 \qquad d^5r' = r^{VI}d\theta^5 + 10r^{IV}d\theta^2 d^3\theta$$

so that this results in

$$d^3\theta - d^3\varphi = \frac{r^{IV}}{r_0}d\theta^3 \qquad d^5\theta - d^5\varphi = \left\{\frac{r^{VI}}{r_0} + \frac{10r^{IV}}{r_0}\left(\frac{r^{IV}}{r_0} + \frac{d^3\varphi}{d\theta^3}\right)\right\}d\theta^5$$

Through the same treatment of the third equation, we obtain

$$d^3\varphi = -\frac{R^{IV}}{R_0}d\beta^3 \qquad d^5\varphi = -\frac{R^{VI}}{R_0}d\beta^5,$$

so that we now know, because $d\vartheta$ can be changed for $d\beta$, all derivations of ϑ and φ from β necessary in further calculations. The successive derivatives of the first equation are

$$\begin{aligned} d^4R &= d^4r \\ d^6R + R_0 d^6\cos(\beta-\varphi) &= d^6r + r_0 d^6\cos(\vartheta-\varphi) \\ d^8R + R_0 d^8\cos(\beta-\varphi) &= d^8r + r_0 d^8\cos(\vartheta-\varphi), \end{aligned}$$

where

$$\begin{aligned} d^6\cos(\beta-\varphi) &= -10(d^3\varphi)^2 & d^6\cos(\vartheta-\varphi) &= -10(d^3\vartheta-d^3\varphi)^2 \\ d^8\cos(\beta-\varphi) &= -56\,d^3\varphi\,d^5\varphi & d^8\cos(\vartheta-\varphi) &= -56(d^3\vartheta-d^3\varphi)(d^5\vartheta-d^5\varphi) \end{aligned}$$

and

$$\begin{aligned} d^4r &= r^{IV}d\vartheta^4 & d^6r &= r^{VI}d\vartheta^6 + 20\,r^{IV}d\vartheta^3 d^3\vartheta \\ d^8r &= r^{VIII}d\vartheta^8 + 56\,r^{VI}d\vartheta^5 d^3\vartheta + 280\,r^{IV}d\vartheta^2(d^3\vartheta)^2 + 56\,r^{IV}d\vartheta^3 d^5\vartheta \\ d^4R &= R^{IV}d\beta^4 & d^6R &= R^{VI}d\beta^6 & d^8R &= R^{VIII}d\beta^8 \end{aligned}$$

so that we obtain the respective formulas through substituting of the values derived above. These can be given the symmetrical form

$$\begin{aligned} R^{IV} &= r^{IV} \\ R^{VI} + 10\frac{R^{IV^2}}{R_0} &= r^{VI} + 10\frac{r^{IV^2}}{r_0} \\ R^{VIII} + 56\frac{R^{VI}R^{IV}}{R_0} + 280\frac{R^{IV^3}}{R_0^2} &= r^{VIII} + 56\frac{r^{VI}r^{IV}}{r_0} + 280\frac{r^{IV^3}}{r_0^2} \end{aligned}$$

from which we see that the magnitudes occurring in them generally represent invariants for any desired parallel curves. The latter must therefore geometrically characterize the common evolute. That this is actually the case is shown when we deduce, from the flattening values through corresponding differentiations, the values $\frac{d^n\rho}{d\varphi^n}$ which represent the radii of curvature of the successive evolutes. I have checked the above formulas through these values.

If we substitute another value for R_o in the equation $R - R_o = f(\vartheta)$, then the equation represents a conchoid with the parallel curve as base. Since now the base curve represents such a conchoid with the infinitely distant parallel curve as base, we obtain the differential quotients of the base curve simply by setting $R_o = \infty$ in the above formulas. The same result is produced by differentiation of the easily verified equations

$$R = r\cos(\vartheta - \varphi) \qquad R' = r'\cos(\vartheta - \varphi) = r\sin(\vartheta - \varphi),$$

where R represents the radius vector in the equation $R = f(\vartheta)$ of the base curve and $\vartheta = \varphi$.

Although the formulas consequently can be applied in the above form also for the base curve, they can be written, by utilizing the distance $\alpha = R_o - r_o$, in the form

$$R^{IV} = r^{IV} \qquad R^{VI} = r^{VI} + \frac{10\,a\,r^{IV^2}}{r_0 R_0}$$

$$R^{VIII} = r^{VIII} + \frac{28\,a\,r^{IV}}{r_0 R_0}(r^{VI} + R^{VI})$$

which is both more convenient for numerical application and can also be used with non-focal surfaces. In this case, the equation $r = f(\vartheta)$ of the meridian curve must be replaced by an equation $f(\xi\eta) = 0$ and the equation $R = f(\vartheta)$ of the parallel curve either permanently or provisionally, depending on the method utilized for grinding, by an equation $f(xy) = 0$ from which follows $a = \eta_o - y_o$, according to the definition of the coordinate system given above (page 70). The formulas thus derived result, for $r_o = \varphi$ and/or $R_o - \varphi$ in

$$\eta^{IV} = -\frac{r^{IV}}{r_0^4} \qquad \eta^{VI} = -\frac{r^{VI}}{r_0^6} \qquad \eta^{VIII} = -\frac{r^{VIII}}{r_0^8}$$

and in the same relations for the parallel curve. The formulas above are first divided by r_o^4 and/or r_o^6 and/or r_o^8 and the last values substituted subsequently. Since all members in the equations so created have finite

values, it is possible to set $r_o = R_o$ in the denominators which produces the formulas

$$y^{IV} = \eta^{IV} \qquad y^{VI} = \eta^{VI} - 10\,a\,\eta^{IV^2}$$
$$y^{VIII} = \eta^{VIII} - 28\,a\,\eta^{IV}(\eta^{VI} + y^{VI})$$

The latter are obtained also through differentiation of the equations

$$\eta - y = a\cos\varphi \qquad x - \xi = a\sin\varphi \qquad y'\cos\varphi = \sin\varphi,$$

of which the last represents the definition of the angle φ formed by the normal with the Y-axis, and the other two can be obtained through projection of the distance a on the coordinate axes.

In order to grind non-focal surfaces with a standard duplex machine without horizontal straight-line guidance, it is necessary to know the differential quotients of the machine curve for the equation in the form $R = f(\beta)$. These are obtained from the differential quotients of the equation $f(xy) = 0$ valid for the respective parallel curve by differentiating, in the same manner as above (page 70), the equations

$$y = R_0 - R\cos\beta \qquad x = R\sin\beta$$

where the distance R_o of the pole of the coordinate system from the vertex of the curve is selected freely. The simplification conditioned by $y'' = 0$ is counteracted by the occurrence, in the values of $d^n y$ and/or $d^n x$, a member $-\binom{n}{2} d^2 R\, d^{n-2}\cos\beta$ and/or $\binom{n}{2} d^2 R\, d^{n-2}\sin\beta$ occurs.

The result is

$$R'' = R_0 \qquad R^{IV} = -y^{IV} R_0' + 5R_0 \qquad R^{VI} = -y^{VI} R_0^{\circ} - 55\,y^{IV} R_0' + 61\,R_0$$
$$R^{VIII} = -y^{VIII} R_0^{\circ} - 140\,y^{VI} R_0^{\circ} - 3486\,y^{IV} R_0' + 280\,y^{IV^2} R_0^{\circ} + 1385\,R_0.$$

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After the respective differential quotients of the machine curve have thus become known, we must first determine the relation of the latter to the machine constants through differentiation of the machine curve. Let us

here employ the abbreviated designation

$$\frac{R-R_0}{R_0}=\varepsilon$$

so that $d^nR = R_o d^n$ in general. The differential quotients $\frac{d^n\varepsilon}{d\alpha^n}$ and/or $\frac{d^n\alpha}{d\beta^n}$ shall be designated as $\varepsilon'\ldots\varepsilon^{IV}\ldots$ and/or $\alpha'\ldots\alpha^{IV}\ldots$ Differentiation produces, for $\alpha = \beta = 0$

$$\frac{d^4\varepsilon}{d\beta^4}=3\,\varepsilon''\,\alpha''^2 \qquad \frac{d^6\varepsilon}{d\beta^6}=15\,\varepsilon''\,\alpha''\,\alpha^{IV}+15\,\varepsilon'''\,\alpha''^3$$

$$\frac{d^8\varepsilon}{d\beta^8}=28\,\varepsilon''\,\alpha''\,\alpha^{VI}+35\,\varepsilon''\,\alpha^{IV^2}+210\,\varepsilon'''\,\alpha''^2\,\alpha^{IV}+105\,\varepsilon^{IV}\alpha''^4.$$

In order to obtain lower numbers for the coefficients and because of the form of the differential quotients of the function representing the crank mechanism, let us employ the abbreviated designations

$$\mathfrak{A}=\frac{R^{IV}}{3R_0} \qquad \mathfrak{B}=\frac{R^{VI}+5\,R^{IV}}{15\,R^{IV}} \qquad \mathfrak{C}=\frac{R^{VIII}-21\,R^{IV}}{105\,R^{IV}}$$

so that we obtain thus

$$\mathfrak{A}=\varepsilon''\,\alpha''^2 \qquad \mathfrak{B}=\frac{1}{3}\left(\frac{\alpha^{IV}}{\alpha''}+\frac{\varepsilon'''\,\alpha''}{\varepsilon''}+1\right)$$

$$\mathfrak{C}=\frac{1}{45}\left(\frac{4\,\alpha^{VI}}{\alpha''}+\frac{5\,\alpha^{IV^2}}{\alpha''^2}+\frac{30\,\varepsilon'''\,\alpha^{IV}}{\varepsilon''}+\frac{15\,\varepsilon^{IV}\,\alpha''^2}{\varepsilon''}-9\right)$$

There are consequently required six successive differentiations of the function $\varphi(\beta)$. For the crank mechanism, this is (cf. p. 39)

$$\varphi(\beta)=1-\cos\beta-\frac{1}{k}(1-\cos\gamma) \qquad \sin\gamma=k\sin\beta.$$

Because of the symmetry, the derivations of odd-number order of the first and those of the even-number order of the second equation vanish for $\beta = 0$. We thus have

$$d^n\varphi(\beta)=-d^n\cos\beta+\frac{d^n\cos\gamma}{k} \qquad n=2,4,6$$

$$d^n\sin\gamma=k\,d^n\sin\beta \qquad n=1,3,5,$$

where

$$d^2\cos\gamma=-d\gamma^2 \qquad d^4\cos\gamma=d\gamma^4-4\,d\gamma\,d^3\gamma$$

$$d^6\cos\gamma=-d\gamma^6+20\,d\gamma^3\,d^3\gamma-10\,(d^3\gamma)^2-6\,d\gamma\,d^5\gamma$$

and

$$d \sin \gamma = d\gamma \qquad d^3 \sin \gamma = -d\gamma^3 + d^3\gamma$$
$$d^5 \sin \gamma = d\gamma^5 - 10\, d\gamma^2 d^3\gamma + d^5\gamma$$

whereas the corresponding first member of these equations occurs only in the derivations of cos β and sin β. Initially, the second equation produces

$$d\gamma = k\, d\beta \qquad d^3\gamma = -k(1-k^2)\, d\beta^3$$
$$d^5\gamma = k(1-k^2)(1-9k^2)\, d\beta^5,$$

and from this results, if

$$t = k(1+k)$$

is set for the sake of brevity

$$\varphi''(\beta) = 1-k \qquad \varphi^{IV}(\beta) = (1-k)(3t-1)$$
$$\varphi^{VI}(\beta) = (1-k)(45k^2t - 15t + 1)$$

In differentiation of the equation of the B-mechanism, we further have

$$d^2 f(\alpha) = f'(\alpha)\, \alpha'' d\beta^2 \qquad d^4 f(\alpha) = \{f'(\alpha)\, \alpha^{IV} + 3 f''(\alpha)\, \alpha''^2\} d\beta^4$$
$$d^6 f(\alpha) = \{f'(\alpha)\, \alpha^{VI} + 15 f''(\alpha)\, \alpha^{IV}\alpha'' + 15 f'''(\alpha)\, \alpha''^3\} d\beta^6$$

and thus obtain

$$\alpha'' = c\frac{\varphi''(\beta)}{f'(\alpha)} \qquad \frac{\alpha^{IV}}{\alpha''} = \frac{\varphi^{IV}(\beta)}{\varphi''(\beta)} - 3\alpha''\frac{f''(\alpha)}{f'(\alpha)}$$
$$\frac{\alpha^{VI}}{\alpha''} = \frac{\varphi^{VI}(\beta)}{\varphi''(\beta)} - 15\alpha^{IV}\frac{f''(\alpha)}{f'(\alpha)} - 15\alpha''^2\frac{f'''(\alpha)}{f'(\alpha)}.$$

In the equation of the sinus mechanism

$$f(\alpha) = \sin\alpha - \operatorname{tg}\omega\,(1-\cos\alpha)$$

is

$$f'(\alpha) = 1 \qquad f''(\alpha) = -\operatorname{tg}\omega \qquad f'''(\alpha) = -1,$$

and in that of the tangential mechanism

$$f(\alpha) = \operatorname{tg}(\alpha+\omega) - \operatorname{tg}\omega$$

there is produced in turn

$$f'(\alpha) = \frac{1}{\cos^2\omega} \qquad f''(\alpha) = \frac{2\operatorname{tg}\omega}{\cos^2\omega} \qquad f'''(\alpha) = \frac{2(1+3\operatorname{tg}^2\omega)}{\cos^2\omega}.$$

In order to examine first what can be accomplished without the utilization of special eccentrics, the function representing the crank mechanism

is also to be employed for $\varphi(\alpha)$ where, however, in order to prevent confusion, the designation $\varkappa$ is to be utilized for the coefficient otherwise represented by k and a number τ corresponding to t shall be defined by the equation

$$\tau = \varkappa^2 + \varkappa$$

In substituting the now deduced values in the equation for $\mathfrak{C}$, there is first to be eliminated $\frac{f''(\alpha)}{f'(\alpha)}$ from the value for $\frac{\alpha^{VI}}{\alpha''}$. The equation so created

$$4\frac{\alpha^{VI}}{\alpha''} + 5\frac{\alpha^{IV^2}}{\alpha''^2} = 4\frac{\varphi^{VI}(\beta)}{\varphi''(\beta)} - 20\frac{\varphi^{IV}(\beta)}{\varphi''(\beta)}\cdot\frac{\alpha^{IV}}{\alpha''} + 25\frac{\alpha^{IV^2}}{\alpha''^2} - 60\,\alpha''^2\frac{f'''(\alpha)}{f'(\alpha)}$$

produces, if the value

$$\frac{\alpha^{IV}}{\alpha''} = 3\mathfrak{B} - 1$$

obtained from the equation for $\mathfrak{B}$ at $\varepsilon''' = 0$ is substituted,

$$4\frac{\alpha^{VI}}{\alpha''} + 5\frac{\alpha^{IV^2}}{\alpha''^2} - 9 = 180\,k^2 t + 225\,\mathfrak{B}^2 - 90\,\mathfrak{B}(1+2t) - 60\,\alpha''^2\frac{f'''(\alpha)}{f'(\alpha)},$$

so that finally, by taking into consideration that

$$\varepsilon'' = C(1-\varkappa) \qquad \frac{\varepsilon^{IV}}{\varepsilon''} = 3\tau - 1$$

there result <u>for a machine with two crank mechanism</u> the formulas

$$\mathfrak{A} = C(1-\varkappa)\alpha''^2 \qquad \mathfrak{B} = t - \alpha''\frac{f''(\alpha)}{f'(\alpha)}$$

in which

$$\mathfrak{C} = 4k^2 t + 5\mathfrak{B}^2 - 2\mathfrak{B}(2t+1) + \alpha''^2\left(\tau - \frac{1}{3} - \frac{4}{3}\frac{f'''(\alpha)}{f'(\alpha)}\right)$$

$$\alpha'' = \frac{c(1-k)}{f'(\alpha)}$$

and the products $C(1-\varkappa)$ and/or $c(1-k)$ at $\varkappa = 1$ and/or $k = 1$ have finite values. With abbreviated designations

$$C_s = \mathfrak{C} + 2\mathfrak{B} - 5\mathfrak{B}^2 \qquad C_t = C_s + 2\mathfrak{B}^2$$

we obtain, for the <u>sinus mechanism</u>,

$$\mathfrak{A} = Cc^2(1-\varkappa)(1-k)^2 \qquad \alpha'' = c(1-k)$$

$$\mathfrak{B} = t + \alpha''\operatorname{tg}\omega \qquad C_s + 4t(\mathfrak{B} - k^2) = \alpha''^2(\tau + 1)$$

and, for the <u>tangential mechanism</u>,

$$\mathfrak{A} = Cc^2\cos^4\omega(1-\varkappa)(1-k)^2 \qquad \alpha'' = c\cos^2\omega(1-k)$$

$$\mathfrak{B} = t - 2\alpha''\operatorname{tg}\omega \qquad C_t + 2k^2(1-k^2) = \alpha''^2(\tau - 3),$$

where the member $8\alpha''^2 \operatorname{tg}^2\omega$ has been eliminated from the last equation.

To begin with, as far as an oscillation of the fourth order is concerned, it will be seen from the formulas that the latter can always be obtained under the assumption that the product $C(1-\gamma)$ has the same sign as $\mathfrak{A}$ as soon as anyone of the four and/or five machine constants can be varied. The greatest possible simplification of the methods results when both ω as well as the coefficients γ and k are made equal to zero and the sinus mechanism is employed. The theoretical calculations are simplified in this manner as much as possible; the two crank mechanisms are represented by eccentrics; and the carriage of the B-mechanism can be replaced by crossed cylinders. Since only the product Cc^2 is determined through the prescribed osculation, there are available an infinite number of solutions among which we can select the best through variation of c and through mathematical checking. On the other hand, if we assign once and for all to c a given value, we obtain a standard non-spherical surface with only one coefficient which can be produced so easily that we can always count on the possibility of procuring such surfaces. The disadvantage that concave surfaces of the second order cannot be produced, is compensated by this method in the simplest possible manner. In order for these surfaces to actually have the properties of such standard surfaces, however, it is necessary that the distance of the parallel curve, selected as machine curve, from the meridian curve of the surface is in a certain ratio to the radius of curvature of the vertex. In order to grind such a surface with prescribed curvature of vertex and prescribed flattening value, it is therefore only necessary to give the corresponding values to the diameter of the abrading surface and to the height of the A-eccentric and to adjust the required distance of the abrading surface from the B-axis. On the

other hand, it is evident that we can also select, in order to accomplish in special cases as much as possible with the simple machine, different parallel curves and/or the base curve as machine curve.

In this manner, it is also possible to achieve <u>osculation of the sixth order</u> with the simplest machine in certain cases. Since $\mathfrak{B} = 0$ in the machine curve, the condition for this is that a parallel curve characterized by this equation can be employed for grinding. That any curve has such a parallel curve is shown by the equation

$$R^{VI} = r^{VI} + \frac{10\,a r^{IV^2}}{r_0 R_0}$$

which is linear in <u>a</u> for $R^{IV} = -5r^{IV}$ and $R_o = a + r_o$ as well as for any value of R^{VI}. Whether an osculation of the sixth order can be obtained with this machine is therefore based on whether the value of <u>a</u> obtained through this equation is mechanically applicable or not.

However, if the angle ω is added as machine constant, then the osculation of the sixth order at any desired machine curve results from two linear equations regardless of whether we employ the sinus or the tangential mechanism. The two equations

$$\mathfrak{B} = c \operatorname{tg} \omega \quad \text{and/or} \quad \mathfrak{B} = -c \sin 2\omega$$

show that c or ω can be selected as desired in the sinus mechanism whereas this is not necessarily the case for c in the tangential mechanism. We can therefore employ the simplest machine if the latter is equipped for a finite angle ω for which only the A-axis needs to be shifted higher. In order to avoid the mechanically disadvantageous negative values of α'', we need have available only two different cylinders which produce one positive and one negative value each of ω when resting on the B-eccentric and which are to be employed correspondingly depending on whether $\mathfrak{B} \gtrless 0$. Osculation of the sixth order is then accomplished by simply giving the two eccentrics the corresponding values.

If the B-mechanism has a carriage so that the coefficient k is available, the respective osculation can be achieved also at $\omega = 0$ when the equation $\mathfrak{B} = t$ produces a real and mechanically employable value of k. The former is the case when $\mathfrak{B} \geq -0.25$ and this condition is satisfied by infinitely many parallel curves among which selection is made in order also to achieve the latter.

Of the equations representing the complete <u>osculation of the eighth order</u>, one is quadratic in α''. Since $\tau + 1$ cannot be made negative but has a minimum value of + 0.75 at a real value of $\varkappa$

$$C_t + 4t(\mathfrak{B} - k^2) > 0$$

is a necessary condition in the sinus mechanism. That the latter cannot be satisfied in general by appropriate selection of the coefficient k will be apparent without detailed discussion. For example, if both C_2 as well as $\mathfrak{B}$ are negative, t must have a negative value in order to satisfy the condition but here the absolute value of this magnitude may not exceed 0.25 so that satisfaction of the condition at a sufficiently large absolute value of C_2 is impossible. <u>It is therefore not possible to achieve osculation of the eighth order at any desired machine curve with the sinus mechanism</u>.

In the tangential mechanism, the necessary condition is

$$\frac{C_t + 2k^2(1 - k^2)}{\tau - 3} > 0,$$

and requires a negative value of C_t at $\varkappa = k = 0$. However, if $C_t \geqq 0$, the numerator can be made negative at $\varkappa = 0$ by a sufficiently large value of k and, on the other hand, the denominator can be made positive at k = 0 through a sufficiently large value of $\varkappa$ so that the condition can always be satisfied through different means. <u>Consequently, osculation of eighth order at any desired machine curve is always possible with the tangential mechanism if the machine contains a variable crank mechanism</u>.

It follows from this that the simplest machine does not correspond in

general to the requirements made by osculation of eighth order but that a carriage is necessary in the B-mechanism. Since this mechanism permits the choice of sinus or tangential mechanism from case to case, the variable crank mechanism is necessary only in the case $C_t > 0 > C_3$ for mathematical reasons because we can make $\varkappa = k = 0$ in all other cases which corresponds to the utilization of eccentrics both in the A- as well as in the B-mechanism. Contrary to this, it may be advantageous for mechanical reasons to employ the crank mechanism also in other cases, in order to influence the value of the other machine constants.

A crank mechanism is more difficult to incorporate in the A-mechanism than in the B-mechanism. To this should be added that its purpose in the first of these mechanisms would be to eventually make $\tau - 3 > 0$ where $\varkappa$, in order not to obtain excessively large values for α'', could be rarely smaller than 1.5 which corresponds to $\tau - 3 = 0.75$. Since the force in the A-mechanism must attack at the crank arm but the length of the couple could represent, in accordance with this, at the most two-thirds of the length of this arm, such a crank mechanism would be mechanically very disadvantageous. However, the force in the B-mechanism can attack the couple so that there is no hesitancy in regard to high values of k from this point of view. Such values have the disadvantage, however, of reducing the maximum value of the angle β and consequently the maximum diameter of the ground surface. However, since a value of $k = 2$ already is $2k^2(1-k^2) = -24$ and further permits $\beta = 30^\circ$, this disadvantage need not be anticipated except in very infrequent cases. It would therefore be generally preferable to make $\varkappa = 0$, i.e., to employ an eccentric in the A-mechanism -- at least as concerns centric osculation.

With $C_3 > 0$ or $C_t < 0$, we must therefore employ the sinus and/or

tangential mechanism and can make k = 0 or, if this procures mechanical advantages, give this coefficient a suitable value. At $C_t > 0 > C_3$, on the other hand, k must first receive such a value that α'' assumes a suitable real value in the tangential mechanism. In all cases, there then result, when the value of α'' has been determined, the machine constants $C_{c\omega}$ from linear equations. If it appears to be of advantage, the latter can subsequently still be influenced through variations of k. To this should be added, moreover, that it will be possible in many cases to favorably influence the values of the coefficient also through suitable selection of the parallel curve. For example, if the parallel curve $\mathfrak{X} = 0$ can be utilized in practice, we have $C_3 = C_t = \mathfrak{C}$, where a real value of α'' is obtained, either in the sinus or in the tangential mechanism, also for k = 0. If $B \geqq -0.25$, we can make $\omega = 0$ from which results $\mathfrak{B} = t$ and

$$C_s + 4t(\mathfrak{B} - k^2) = C_t + 2k^2(1 - k^2)$$

so that we obtain again a real value of α'' either in the sinus or in the tangential mechanism. In contrast to this, we would have to solve an equation of the fourth order if we were to assign a given finite value to ω.

Although consequently the utilization of <u>special cylinders</u> to achieve centric osculation of the eighth order would seem to be superfluous, the latter are indispensable either in certain special cases to be treated further below or else furnish certain advantages also in the general case. Initially, there arises the question whether a curved-line guidance and/or an eccentric would not be able to accomplish a more exact positive guidance than the crank mechanism and it is further possible, with the employment of special cylinders, to make do in all cases, even with the simplest machine, without a carriage in the B-mechanism. Condition for this is evidently that the equation for $\mathfrak{C}$ is linear in a machine constant introduced by such a cylinder. The latter cannot be contained in ε''' because

α^{IV} contains either no machine constant when the constant [?] contained in ε''' would be determined already by $\mathfrak{B}$ or else there results a quadratic equation. On the other hand it will seem obvious that both ε^{IV} as well as α^{VI} may contain the respective constant. It follows from this that when the special cylinder is utilized in the A-mechanism, differentiation four times of the respective function is sufficient whereas the B-mechanism would require one of six times. When employing curves of the second order for curved-line guidance, more complicated expressions result for the differential quotients than by the employment of them and/or their parallel curves in an eccentric mechanism. Curved-line guidance should therefore be considered only for the A-mechanism but the eccentric also for the B-mechanism. This last combination has the advantage of making the B-carriage superfluous to which should be added that one and the same cylinder is utilized for the grinding of surfaces with different curvature of vertex.

If we are merely concerned to replace the crank mechanism by an eccentric in those cases where osculation of the eighth order requires a value of $k \neq 0$ and where c and k are consequently given and necessarily

$$\varphi(\beta) = 1 - \cos\beta - \frac{1}{k}(1 - \cos\gamma) \qquad \sin\gamma = k \sin\beta$$

we must set $e = |k|$ and $k_1 = k$ in the formulas deduced above (cf. p. 49) from which is obtained

$$\rho = \frac{a_1(1 - k^2)}{k} \qquad r_0 = \frac{\rho}{1 + k}$$

Since $a_1 = cE_0$ and $\alpha'' = c\,(1-k)$, these expressions can also be written in the form

$$\frac{\rho}{E_0} = \frac{\alpha''(1 + k)}{k} \qquad \frac{r_0}{E_0} = \frac{\alpha''}{k}$$

It follows from this that, at $0 \geqslant k > -1$, ρ has the opposite sign from α'', so that the eccentric must act downward at a positive value of α'' but is able to act upward otherwise and that the axis of the eccentric must be

located below or above the vertex of the curve at positive α'', depending on whether k has a positive or a negative value. However, as will be seen from the foregoing, since k does not need to have a negative value in the cases where this constant cannot be made equal to zero, we can always employ an eccentric acting upward and whose axis is located below the vertex of the curve. Whether we will employ the curve of the second order or a parallel curve of the latter will be decided on the basis of mechanical considerations. However, the magnitudes ρ and r_o occurring in the formulas always refer to the conical section itself.

For the successive differentiations, the following form of the equation is best suitable. We set

$$l = a\varphi(\beta) \qquad a = \rho - r_0 \qquad k = \frac{a}{\rho},$$

where a then represents the distance of the center of curvature of the vertex from the axis of the eccentric and is considered as positive when the former is located below the latter. This results in

$$(1-e^2)\varphi(\beta) = \left(1 + \frac{e^2(1-k)}{k}\right)(1-\cos\beta) - \frac{1}{k}(1-u) \qquad u^2 = 1 - e^2\sin^2\beta,$$

in which e^2 may also have a negative value. In the values of the differential quotients $\frac{d^n\varphi(\beta)}{d\beta^n}$ no other derivations occur except $\frac{d^n \cos\beta}{d\beta^n}$ and $\frac{d^n u}{d\beta^n}$ where those of odd-number order vanish because of the symmetry. The second equation produces

$$d^2u = -e^2(d\sin\beta)^2 \qquad d^4u + 3(d^2u)^2 = -4e^2 d\sin\beta d^3\sin\beta$$
$$d^6u + 15\,d^2u d^4u = -e^2\{6\,d\sin\beta d^5\sin\beta + 10(d^3\sin\beta)^2\},$$

and, when substituting these values in the derivations of the first equation, the right side becomes divisible by $1-e^2$ without residue. We obtain in this manner

$$\varphi''(\beta) = 1 \qquad \varphi^{IV}(\beta) = \frac{3e^2}{k} - 1$$

$$\varphi^{VI}(\beta) = \frac{45e^4}{k} - \frac{15e^2}{k} + 1.$$

We will see instantly that the equation for $\mathfrak{B}$ is linear in $\frac{e^2}{k}$ and that, after this magnitude has been determined, the equation for $\mathfrak{C}$ is linear in e^2. We obtain first, for $\varepsilon''' = 0$, by replacing $\frac{\alpha^{IV}}{\alpha''}$ with $3\mathfrak{B} - 1$,

$$\frac{1}{3}\left(\frac{\varphi^{IV}(\beta)}{\varphi''(\beta)} + 1\right) = \frac{e^2}{k}$$

$$4\frac{\varphi^{VI}(\beta)}{\varphi''(\beta)} - 20\frac{\varphi^{IV}(\beta)}{\varphi''(\beta)}\frac{\alpha^{IV}}{\alpha''} + 25\frac{\alpha^{IV^2}}{\alpha''^2} = 180\frac{e^4}{k} + 225\mathfrak{B}^2 - 90\mathfrak{B}\left(1 + \frac{2e^2}{k}\right) + 9$$

and, after substitution of these values, for the sinus mechanism

$$\mathfrak{A} = Cc^2(1-\varkappa) \qquad \mathfrak{B} = \frac{e^2}{k} + c\,\mathrm{tg}\,\omega$$

$$C_s + \frac{4e^2}{k}(\mathfrak{B} - e^2) = c^2(\tau + 1)$$

and/or for the tangential mechanism

$$\mathfrak{A} = Cc^2\cos^4\omega(1-\varkappa) \qquad \alpha'' = c\cos^2\omega$$

$$\mathfrak{B} = \frac{e^2}{k} - 2\alpha''\,\mathrm{tg}\,\omega \qquad C_t + \frac{2e^2}{k}\left(\frac{e^2}{k} - 2e^2\right) = \alpha''^2(\tau - 3),$$

if a B-eccentric is employed whose fundamental curve represents a conical section or the parallel curve of the former. A negative value of e^2 corresponds to an ellipse whose shorter axis is vertical in the zero position and whose semi-axes are obtained in the manner indicated. If k has a negative value, this is the case also either with α or ρ in which α'' is negative, i.e., the eccentric acts downward. The latter is not possible in the simplest machine and the former is not advantageous. However, a negative value of k is not necessary to make the sinus mechanism applicable also for the cases $C^s \lesseqgtr 0$ at $\varkappa = 0$. If we write this last equation

$$C_s + 4\mathfrak{B}(\mathfrak{B} - c\,\mathrm{tg}\,\omega) - c^2 = \frac{4e^4}{k},$$

it is then evident, after c has received a suitable positive value, that we can select ω so that $k > 0$. Only in those cases when in this manner a mechanically disadvantageous value of ω would result, would it be preferable

to apply the eccentric and/or a curved-line guidance in the A-mechanism. In the equations deduced for a machine with two crank-mechanisms, $1 - x$ need then be replaced only by $\varphi''(\alpha)$ and τ by $\frac{1}{3}\left\{\frac{\varphi^{IV}(\alpha)}{\varphi''(\alpha)} + 1\right\}$. However, since here a crank mechanism in the B-mechanism is not excluded, it is preferable to designate the coefficient k occurring in the equation of the eccentric and/or the curve-line guidance by k_e. If an A-eccentric is utilized in which the fundamental curve represents a conical section or the parallel curve of the former, we must then set $\varkappa = 0$ and replace τ by $\frac{e^2}{k_e}$ in the equations valid for a machine with two crank mechanisms. The osculation of the eighth order can then in all cases be obtained with the sinus mechanism, without the necessity for giving k_e a negative value. It follows from this that the simplest machine, under the conditions specified above, for the osculation of the sixth order always makes possible the osculation of the eighth order when employing such an eccentric.

This is also true of the corresponding curved-line guidances, although the expressions then do not become as simple. For a cylinder linked with the axis, the equations deduced above (p. 43) must be differentiated, by setting $\zeta = a\varphi(\beta)$ in the form

$$(1 - e^2\cos^2\beta)\varphi(\beta) = (1 - w\cos\beta)(1 - \cos\beta) - \frac{1}{k}(1 - u) \qquad u^2 = 1 - v\sin^2\beta,$$

in which

$$w = \frac{e^2(1-k)}{k} \qquad v = k^2 - e^2(1-k)^2$$

We thus obtain

$$(1 - e^2)\varphi''(\beta) = 1 - w + \frac{1}{k}\frac{d^2u}{d\beta^2}$$

$$(1 - e^2)\varphi^{IV}(\beta) = 6e^2\varphi''(\beta)\frac{d^2(\cos^2\beta)}{d\beta^2} - 1 + w + 6w\left(\frac{d^2\cos\beta}{d\beta^2}\right)^2 + \frac{1}{k}\frac{d^4u}{d\beta^4},$$

where the derivations of u have the already deduced form. After the respective values have been inserted, the right side of the equations becomes divisible without residue both by $1 - e^2$ and by $1 - k$ and this results, when

we set α for β and k_e for k, in

$$\varphi''(\alpha) = 1 - k_e \qquad \frac{1}{3}\left(\frac{\varphi^{IV}(\alpha)}{\varphi''(\alpha)} + 1\right) = k_e^2 + k_e + \frac{e^2(1-k_e)^2}{k_e},$$

and these values must then apply in the manner indicated above in the equations valid for a machine with two crank-mechanisms.

If the cylinder is fixed to the carriage, the corresponding equations

$$\varphi(\beta) = 1 - \cos\beta + \frac{1}{kq}(1-u) \qquad u^2 = 1 + k^2 q \sin^2\beta$$

produced in the same manner

$$\varphi''(\alpha) = 1 - k_e \qquad \frac{1}{3}\left(\frac{\varphi^{IV}\alpha}{\varphi''(\alpha)} + 1\right) = k_e^2 + k_e + \frac{k_e^2 e^2}{1-k_e}$$

and k_e can be established in both cases by taking into account mechanical advantages.

The application of evolvent eccentrics to obtain osculation of the eighth order will hardly be practical in the general case. An evolvent of the first order would offer no advantages and an evolvent of the second order does presuppose the grinding of two special cylinders. However, with the utilization of the latter, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ would then be determined respectively by ε'', ε''', ε^{IV} so that the three variable machine constants would all be obtained through the shape of the A-eccentric. Theoretically ideal, this method of making the simplest machine applicable to all cases has the only disadvantage of being expensive.

On the other hand, <u>duplex cylinders</u> can be employed to considerable advantage because they can be made in the machine itself. The equation of the machine curve is that of the corresponding triplex curve when utilizing a duplex cylinder. If the cylinder is to perform as a B-eccentric, we then have the general equation

$$\frac{R - R_0}{R_0} = C\varphi(\alpha) \qquad f(\alpha) = cD$$

$$D = \psi(\beta) + k_1\varphi(\gamma) \qquad f(\gamma) = c_1\varphi(\beta),$$

in which we will make

$$\varphi(\alpha) = 1 - \cos\alpha \qquad \psi(\beta) = \varphi(\beta) = 1 - \cos\beta \qquad \varphi(\gamma) = 1 - \cos\gamma$$

$$f(\alpha) = \sin\alpha - \operatorname{tg}\omega(1 - \cos\alpha) \qquad f(\gamma) = \sin\gamma - \operatorname{tg}\omega_1(1 - \cos\gamma)$$

since this concerns the application of the simplest machine. Differentiation produces initially

$$\alpha'' = cD'' \qquad \frac{\alpha^{IV}}{\alpha''} = \frac{D^{IV}}{D''} - 3\alpha''\frac{f'''(\alpha)}{f'(\alpha)}$$

$$\frac{\alpha^{VI}}{\alpha''} = \frac{D^{VI}}{D''} - 15\alpha^{IV}\frac{f''(\alpha)}{f'(\alpha)} - 15\alpha''^2\frac{f'''(\alpha)}{f'(\alpha)},$$

in which $\frac{d^2D}{d\beta^2}\ldots$ is designated by D'' ... and in which

$$D'' = \psi''(\beta) \qquad D^{IV} = \psi^{IV}(\beta) + 3k_1\varphi''(\gamma)\gamma''^2$$

$$D^{VI} = \psi^{VI}(\beta) + 15k_1\varphi''(\gamma)\gamma''\gamma^{IV}$$

Since ω_1 occurs only in the value

$$\gamma^{IV} = c_1(3c_1\operatorname{tg}\omega_1 - 1)$$

it is apparent already at this stage of the calculation that the equation for linear $\mathfrak{C}$ in the new machine constant is $\operatorname{tg}\omega_1$. Calculation takes place in the same manner as above for a machine with two crank-mechanisms and produces

$$\mathfrak{A} = Cc^2 \qquad \mathfrak{B} = c\operatorname{tg}\omega + k_1c_1^2$$

$$C_s + 4k_1c_1^2(\mathfrak{B} - c_1\operatorname{tg}\omega_1) = c^2,$$

and these formulas are consequently valid for a machine with sinus mechanism in which a duplex cylinder ground in the same machine is utilized as B-eccentric and where the A-mechanism is represented by a standard eccentric. It is obvious that the simplest machine is sufficient in all cases, provided only one given positive and one given negative value of ω and/or ω_1 is possible because we then have available also the sign of the three coefficients c, c_1, k_1. The cylinder must be ground with one plane and the radius of curvature of the vertex ρ_o can be selected arbitrarily. If $C_o\omega_o c_o$ represents the machine constants to be utilized in the grinding of the latter, then we

must make

$$C_0 = \frac{ck_1E_0}{\rho_0} \qquad \omega_0 = \omega_1 \qquad c_0 = c_1$$

and, in the utilization of the duplex cylinder, the distance of the center of curvature of the vertex from the B-axis equal to cE_0 and in that case, when this magnitude is positive, this point will be located below the axis.

If a duplex cylinder is utilized in the A-mechanism, then the machine curve has the equation

$$\frac{R-R_0}{R_0} = C\{\varphi(\alpha) + k_1\varphi(\gamma)\} \qquad f(\gamma) = c_1\varphi(\alpha) \qquad f(\alpha) = c\varphi(\beta),$$

and we will here assume the same simple functions as above. Since $\frac{d^2\gamma}{d\beta^2} = 0$, there occurs, as will be seen from the formulas deducted above (p. 77), a number containing γ only in the value for $\frac{R^{VIII}}{R}$, i.e.

$$+ 35Ck_1\varphi''(\gamma)\left(\frac{d^4\gamma}{d\beta^4}\right)^2,$$

in which

$$\frac{d^4\gamma}{d\beta^4} = 3c_1c^2$$

In the value for $\mathfrak{C}$, this number is divided by $315\ Cc^2$ and consequently equal to k_1, c_1^2, c^2. This produces therefore the equations

$$\mathfrak{A} = Cc^2 \qquad \mathfrak{B} = c\ \mathrm{tg}\ \omega \qquad C_s = c^2(1 + k_1c_1^2),$$

which furnish the machine constants in the easiest manner. That the value of k_1 becomes negative at $C_s < 0$, produces no disadvantage in the A-mechanism. Since ω_1 actually only occurs in the derivations of higher than the eighth order, this angle can therefore be made equal to zero when grinding the cylinder. In the case that the latter is to be used as eccentric, the detailed instructions for its production have already been given above (p. 65). However, if the duplex cylinder is to be utilized for the purpose of curved-line guidance, then the machine curve in the production of the latter must be a pseudo-duplex curve. If we write the equation of this curve as

$$\frac{R-\rho_0}{\rho_0} = C_0\varphi(\alpha) + C_1\varphi(\beta) \qquad f(\alpha) = c\varphi(\beta),$$

we then have $\rho_0 C_1 = R_0 C \qquad C_0 = k_1 C_1 \qquad c = c_1$

and can arbitrarily select ρ_o. When utilizing the cylinder, the roller must have the same diameter as in the production of the abrading cylinder and the distance of the roller axis from the A-axis in the initial position must be equal to ρ_o. Since $\varphi(\alpha) = 1 - \cos\alpha$ the cylinder can therefore be ground with the simplest duplex machine in accordance with the instructions given above (p. 63)

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Among the special cases, we encounter first the case $r_o = 0$ which can be realized for convex surfaces not intersected by the evolute. In regard to optical surfaces, no need appears to exist so far which would lead to this special case. However, it may be of advantage, in the production of duplex eccentrics, to have the possibility, in arbitrary selection of the radius of curvature of the vertex, to make the latter equal to zero. Thus $R_o = 0$ and we have to use $A = \frac{R^{IV}}{3}$ in the above calculations instead of . The first equation of the machine curve is multiplied by R_o and the product $R_o C$ receives a finite value but the calculations remain unchanged otherwise.

As can be seen from the equations (p. 74), $\eta^{IV} = R^{IV}$ in the equation of the fundamental curve of the cylinder ground with one plane whereas r^{VI} and r^{VIII} have infinitely large values. However, this is without importance since only the machine curve is of influence in the production as well as in the utilization of the eccentric.

The case $\eta^{IV} = 0$ permits only one solution at $r^{VI} \neq 0$. Since $R^{IV} = 0$ and $R^{VI} = r^{VI}$ both for the base curve as well as for each parallel curve, we must have $\alpha'' = 0$ or $\varepsilon'' = 0$. If the former is the case, then also $R^{VI} = 0$ whereas in the latter case both R^{VI} as well as R^{VIII} can have finite

values at $\varepsilon''' \neq 0$. The resulting condition $\varepsilon''' \neq 0$ at $\varepsilon'' = 0$ can be complied with by an evolvent eccentric of the first order if the A-axis passes through that point of the evolute circle in which the tangent is in the initial position perpendicular to the plane linked to the A-carriage and resting on the eccentric. We then have

$$\varphi(\alpha) = \alpha - \sin\alpha$$

and consequently, for $\alpha = 0$

$$\varepsilon'' = \varepsilon^{IV} = 0 \qquad \varepsilon''' = C,$$

where C represents the ratio of the radius of the evolute circle to R_o. We then obtain first

$$\frac{d^6\varepsilon}{d\beta^6} = 15\,\varepsilon'''\alpha''^3 \qquad \frac{d^8\varepsilon}{d\beta^8} = 210\,\varepsilon'''\alpha''^2\alpha^{IV}$$

and from this

$$\frac{R^{VI}}{15\,R_0} = Cc^3 \qquad \frac{R^{VIII} + 14\,R^{VI}}{42\,R^{VI}} = c\,\mathrm{tg}\,\omega.$$

for the simplest machine.

In the case $\gamma^{IV} = r^{VI} = 0$, this is true also for the parallel curve and for the base curve and for them also $R^{VIII} = r^{VIII}$. Osculation of the eighth order requires either $\varepsilon'' = \varepsilon''' = 0$ at $\varepsilon^{IV} \neq 0$ or $\alpha'' = 0$ at $\alpha^{IV} \neq 0$. Such a condition can be complied with only by a special cylinder whose evolute edge coincides with the respective machine axis. If the fundamental curve of the cylinder is a conical section and/or the parallel curve of one, we then must make, when using the cylinder as eccentric, a = 0 and $\varphi(\beta) = \frac{l}{\rho}$ from which we obtain

$$\varphi(\beta) = \frac{e^2(1-\cos\beta) - 1 + u}{1 - e^2} \qquad u^2 = 1 - e^2\sin^2\beta$$

and, if the conical section is to be utilized as a guidance curve linked to the axis, we obtain in the same manner

$$\varphi(\beta) = \frac{-e^2\cos\beta(1-\cos\beta) - 1 + u}{1 - e^2\cos^2\beta} \qquad u^2 = 1 + e^2\sin^2\beta,$$

However, if the guidance curve is to be rigidly linked to the carriage, we must make a = ρ which results in

$$\varphi(\beta) = 1 - \cos\beta + \frac{1-u}{q} \qquad u^2 = 1 + q\sin^2\beta$$

In all three cases, we obtain

$$\varphi''(\beta) = 0 \qquad \varphi^{IV}(\beta) = 3e^2,$$

for $\beta = 0$ where e^2 can also have a negative value. If the cylinder is utilized as eccentric in the B-mechanism, we then have $\alpha^{IV} = 3ce^2$ and obtain

$$\frac{R^{VIII}}{315R_0} = Cc^2e^4,$$

for the simplest machine whereas, if it functions in one way or another in the A-mechanism, $\varepsilon^{IV} = 3Ce^2$ and we then obtain

$$\frac{R^{VIII}}{315R_0} = Cc^4e^2$$

The former manner of application has the advantage among others that any given cylinder possibly on hand can be utilized which depends in the latter on the sign of R^{VIII}.

A duplex cylinder will also comply with the condition when the machine curve of the surface to be ground is given the form

$$\frac{R-R_0}{R_0} = C\varphi(\alpha) \qquad f(\alpha) = c_1\varphi(\gamma) \qquad f(\gamma) = c\varphi(\beta)$$

where α can be interchanged with γ. This results in

$$\frac{R^{VIII}}{315R_0} = Cc_1^2c^4,$$

for the simplest machine regardless of whether the cylinder is utilized in the A- or in the B-mechanism. Here also exists the same difference between these methods of application.

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In regard to osculation of the eighth order, we now need only treat the non-focal surfaces. In the machine curve

$$y = C\varphi(\alpha) \qquad f(\alpha) = c\varphi(x)$$

it will be simplest to replace the function $\varphi(x)$ by a function $\varphi(\beta)$ with the addition of an equation $x = a_0 f(\beta)$. We then need merely form the differential quotients $\frac{d^n y}{d\beta^n}$ which can then be used in the above calculations instead of the values R^{IV} ... so that the calculations remain unchanged in all details. The simplest values are obtained when $f(\beta) = \sin \beta$. Differentiation first results in

$$d^4y = y^{IV}dx^4 \qquad d^6y = y^{VI}dx^6 + 20\,y^{IV}dx^3d^3x$$
$$d^8y = y^{VIII}dx^8 + 56\,y^{VI}dx^5d^3x + 280\,y^{IV}dx^2(d^3x)^2 + 56\,y^{IV}dx^3d^5x,$$

from which result the values

$$\frac{d^4y}{d\beta^4} = y^{IV}a_0^4 \qquad \frac{d^6y}{d\beta^6} = y^{VI}a_0^6 - 20\,y^{IV}a_0^4$$
$$\frac{d^8y}{d\beta^8} = y^{VIII}a_0^8 - 56\,y^{VI}a_0^6 + 336\,y^{IV}a_0^4$$

We can then write, as equation of the machine curve

$$\frac{y}{a_0} = C\varphi(\alpha) \qquad f(\alpha) = c\varphi(\beta)$$

and make $\mathfrak{A} = \frac{1}{3a_0}\frac{d^4y}{d\beta^4}$, after which all deductions receive identically the same form as for standard surfaces.

This can be accomplished also when utilizing a pseudo-duplex curve as machine curve. The above deductions are actually valid generally on conditions that $\varepsilon, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ is defined by the equations

$$\varepsilon = C\varphi(\alpha) \qquad f(\alpha) = c\varphi(\beta)$$
$$\mathfrak{A} = \frac{1}{3}\frac{d^4\varepsilon}{d\beta^4}, \qquad \mathfrak{B} = \frac{\frac{d^6\varepsilon}{d\beta^6} + 5\frac{d^4\varepsilon}{d\beta^4}}{15\frac{d^4\varepsilon}{d\beta^4}} \qquad \mathfrak{C} = \frac{\frac{d^8\varepsilon}{d\beta^8} - 21\frac{d^4\varepsilon}{d\beta^4}}{105\frac{d^4\varepsilon}{d\beta^4}}$$

The equation of a pseudo-duplex curve can be expressed in polar coordinates by

$$\frac{R-R_0}{R_0} - C_1\psi(\beta) = \varepsilon$$

and the latter can consequently be generally utilized for obtaining an osculation of the eighth order, provided C_1 and the machine constants possibly contained in the function $\psi(\beta)$ can be selected arbitrarily. With standard surfaces, however, the method would become only more complicated by this. However, when we are concerned with non-focal surfaces, it offers a means by which we can avoid the otherwise necessary horizontal straight-line guidance. In that case, we must have $C_1\psi''(\beta) = 1$ for the vertex and it is preferable to make $\psi(\beta) = 1 - \cos\beta$ which corresponds to the utilization of a standard eccentric. Since consequently $C_1 = 1$, we obtain

$$\mathfrak{A} = \frac{R^{IV} + R_0}{3R_0} \qquad \mathfrak{B} = \frac{R^{VI} + 5R^{IV} + 4R_0}{15(R^{IV} + R_0)}$$

$$\mathfrak{C} = \frac{R^{VIII} - 21R^{IV} - 20R_0}{105(R^{IV} + R_0)}.$$

The necessary compound duplex machine required for production in the general case can be constructed according to the type of the simplest machine and makes possible, if it is equipped with two B-eccentrics, the application of special cylinders both in the A- and in the B-mechanism. However, if we make $\varphi(\beta) = \psi(\beta)$, in order to eliminate the second B-eccentric, we may possibly need to use special cylinders in the A-mechanism. Finally, at $C_s > 0$, a standard duplex machine can be utilized for grinding if we make $\varphi(\alpha) = 1 - \cos\alpha$ and the angle δ is determined by the method indicated above (p. 62).

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Centric osculation of the eighth order can consequently be obtained in all cases. If the B-mechanism represents a variable crank-mechanism with

straight-line guidance, special cylinders will then be required only when the surface to be ground has a contact of at least the fourth order with a sphere. The simplest machine characterized by A-eccentrics and crossed cylinders is always adequate, although special cylinders may be necessary in certain cases. A circular evolvent eccentric, however, only of the first order, is required only when the surface has a contact of the fourth but not of the sixth order with a sphere. In all other cases, the special cylinders can be ground arbitrarily either in the simplest duplex machine or in the machine described for the grinding of surfaces of the second degree. Non-focal surfaces can be ground, without the horizontal straight-line guidance, by using a compound duplex machine and, in certain cases, even with the simplest machine.

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Eccentric Osculation

If it has been prescribed that the meridian curve of the ground surface shall pass through a point given in relation to the vertical apex and at a finite distance from the latter and that the normal shall have a given inclination toward the axis in this point, then an eccentric osculation of the first order has been prescribed which passes over into one of the second order if the radius of curvature in the given point is also given. We then need first to determine the respective values valid for the machine curve. If the latter constitutes a parallel curve and if the equation of the meridian curve of the surfaces is given in Cartesian coordinates where the X-axis is to coincide with the axis of symmetry and the Y-axis with the vertical tangent, then the magnitudes φ N M are to be determined through the equations defining them as indicated above (p. 47)

$$\operatorname{tg}\varphi = \frac{dx}{dy} \qquad N = \frac{y}{\sin\varphi} \qquad M = x + N\cos\varphi$$

For the radius of curvature ρ, we have

$$\frac{1}{\rho} = \cos^3\varphi \frac{d^2x}{dy^2}.$$

If, on the other hand, the equation of the meridian curve of the surface exists in polar coordinates in which the coordinate system is to be determined through the relations

$$x = r_0 - r\cos\theta \qquad y = r\sin\theta$$

to the rectangular coordinate system defined above, we then obtain for the equation of curve $r = f(\theta)$, the same magnitudes from the equations

$$\operatorname{tg}(\theta - \varphi) = \frac{r'}{r} \qquad N\sin\varphi = r\sin\theta$$

$$(M - r_0)\sin\varphi = r\sin(\theta - \varphi) \qquad \frac{1}{\rho} = \frac{\cos^3(\theta - \varphi)}{r^3}(r^2 + 2r'^2 - rr'').$$

The corresponding values N_a M_a ρ_a of the parallel curve located at the distance a are N + a, etc. From these, we obtain the coordinates Rβ and the differential quotients R'R'' of the equation R = f(β) of the machine curve through the above relations by beginning with the equation

$$\operatorname{tg}\beta = \frac{N_a \sin\varphi}{N_a \cos\varphi - M_a + R_0}$$

If the machine curve is to be represented in Cartesian coordinates, then the relations first indicated above produce the coordinates and differential quotients at the corresponding point of the former.

If a base curve R = f(β) is to be utilized as machine curve, we have β = φ and obtain R R' R'' from the equations

$$R = N - (M - R_0)\cos\varphi \qquad R' = (M - R_0)\sin\varphi \qquad R'' = \rho - R,$$

of which the first is obtained by projection of the radius vector on the normal whereas the other two are derived most simply from those valid for a parallel curve. Since the base curve represents a conchoid with the infinitely distant parallel curve as base, the differential quotients R' R'' have the same values as for this curve so that we need only to make a = ∞ and β = φ in the respective equations. If R_a represents the infinitely large radius vector of the parallel curve, we then have

$$R'\cos(\beta - \varphi) = R_a \sin(\beta - \varphi) = (M - R_0)\sin\varphi,$$

where M - R_o has the same value for the infinitely distant parallel curve as for the fundamental curve. Since further the radius of curvature of the parallel curve is R_a - R + ρ, there then results generally for such a curve

$$R'' = R_a\{1 + 2\operatorname{tg}^2(\beta - \varphi)\} - \frac{R_a^2}{(R_a - R + \rho)\cos^3(\beta - \varphi)}.$$

This expression is valid for any desired parallel curve if R represents the radius vector of the conchoid of the latter defined through the relation R = R_a - a. The right side is now to be converted into a fraction

and the denominator and the numerator are then to be divided by R_a whereupon we can make $\frac{1}{R_a} = 0$ and $\beta - \varphi = 0$. The numerator here contains the member

$$R_a\{\cos^3(\beta-\varphi)[1+2\,\mathrm{tg}^2(\beta-\varphi)]-1\},$$

which is, however, equal to zero because $R_a \sin(\beta - \varphi)$ has a finite value so that the simple relation above results.

After we have thus determined the respective magnitudes of the machine curves, the elimination necessary for the determination of the machine constants can be performed. In the equation of the true duplex curve

$$\frac{R-R_0}{R_0} = C\varphi(\alpha) \qquad f(\alpha) = c\varphi(\beta)$$

$R_o = \rho_o + a$ and/or $R_o = \rho_o$, depending on whether the latter represents a parallel curve or the base curve of the meridian curve of the ground surface. For the sake of brevity, let us set

$$R - R_0 = K \qquad \varphi(\beta) = B$$

Two successive differentiations produce

$$\frac{R'}{R_0} = C\varphi'(\alpha)\alpha' \qquad f'(\alpha)\alpha' = cB'$$

$$\frac{R''}{R_0} = C\{\varphi''(\alpha)\alpha'^2 + \varphi'(\alpha)\alpha''\} \qquad f''(\alpha)\alpha'^2 + f'(\alpha)\alpha'' = cB'',$$

from which we obtain first, through elimination of C and c,

$$\frac{R'}{K} = \frac{\varphi'(\alpha)}{\varphi(\alpha)}\alpha' \qquad \frac{B'}{B} = \frac{f'(\alpha)}{f(\alpha)}\alpha'$$

$$\frac{R''}{R'} = \frac{\varphi''(\alpha)}{\varphi'(\alpha)}\alpha' + \frac{\alpha''}{\alpha'} \qquad \frac{B''}{B'} = \frac{f''(\alpha)}{f'(\alpha)}\alpha' + \frac{\alpha''}{\alpha'}$$

and then, through elimination α' and α''

$$\frac{R'}{K} = \frac{B'}{B}\cdot U \qquad \frac{R''}{R'} = \frac{B''}{B'} + \frac{B'}{B}\cdot V$$

where

$$U = \frac{\varphi'(\alpha)f(\alpha)}{\varphi(\alpha)f'(\alpha)} \qquad V = \frac{f(\alpha)}{f'(\alpha)}\left(\frac{\varphi''(\alpha)}{\varphi'(\alpha)} - \frac{f''(\alpha)}{f'(\alpha)}\right)$$

If the function B is known, U and V are thus determined by R, R' and the three machine constants C c ω can be utilized for obtaining the eccentric osculation of the second order if it is possible to calculate the angles ω and α from the values of U and V and if these angles receive mechanically applicable values. The next task is consequently to investigate these functions for the different mechanisms. Since an eccentric is to be preferred in the A-mechanism simply for mechanical reasons and since the calculations here would obviously be purposeless when utilizating special eccentrics, it shall be here assumed that a standard eccentric represents the A-mechanism.

For the sinus mechanism, we therefore have

$$\varphi(\alpha) = 1 - \cos\alpha \qquad f(\alpha) = \sin\alpha - \operatorname{tg}\omega\,(1 - \cos\alpha).$$

By taking into account that

$$\sin\alpha = \frac{1 - \cos^2\alpha}{\sin\alpha} \qquad \cos\alpha - \sin\alpha \operatorname{tg}\omega = \frac{\cos(\alpha + \omega)}{\cos\omega}$$

we obtain

$$f(\alpha) = \frac{1 - \cos\alpha}{\sin\alpha}\left(1 + \frac{\cos(\alpha + \omega)}{\cos\omega}\right),$$

whereas the expression

$$f(\alpha) = \frac{\sin(\alpha + \omega)}{\cos\omega} - \operatorname{tg}\omega$$

is easier for the differentiation. The latter produces

$$\varphi'(\alpha) = \sin\alpha \qquad f'(\alpha) = \frac{\cos(\alpha + \omega)}{\cos\omega}$$

$$\varphi''(\alpha) = \cos\alpha \qquad f''(\alpha) = -\frac{\sin(\alpha + \omega)}{\cos\omega}$$

and consequently

$$\frac{\varphi''(\alpha)}{\varphi'(\alpha)} - \frac{f''(\alpha)}{f'(\alpha)} = \cot\alpha + \operatorname{tg}(\alpha + \omega) = \frac{\cos\omega}{\sin\alpha\cos(\alpha + \omega)},$$

so that there finally results

$$U = 1 + \frac{\cos\omega}{\cos(\alpha + \omega)} \qquad V = \frac{U(U - 1)}{1 + \cos\alpha}$$

Since

$$\frac{1}{U-1} = \cos\alpha - \sin\alpha \operatorname{tg}\omega$$

the values of the angles α and ω result from the equations

$$\cos\alpha = \frac{U(U-1)-V}{V} \qquad \operatorname{tg}\omega = \cot\alpha - \frac{1}{(U-1)\sin\alpha}.$$

Since both the absolute value of ω as well as that of $\alpha + \omega$ must be smaller than $\frac{\pi}{2}$, it follows from the value U that we must get $U > 1$ from which follows in turn $V > 0$. The condition $1 > \cos\alpha > -\lambda^2$ where $\lambda^2 < 1$, can then be written in the form

$$2V > U(U-1) > V(1-\lambda^2) > 0$$

At a high negative value of ω, an angle of $\alpha > \frac{\pi}{2}$ is actually mechanically possible, although it is obviously more advantageous if we can make $\lambda^2 = 0$. Since further the absolute magnitude of the angle ω may not exceed a certain value ω_m for technical reasons, there is added to this also the condition

$$U \geq 1 + \cos\omega_m$$

from which it will be clearly seen that a carriage in the B-mechanism permits smaller values of U than the method of the crossed cylinders.

For the <u>tangent mechanism</u> with standard A-eccentric, there is valid

$$\varphi(\alpha) = 1 - \cos\alpha \qquad f(\alpha) = \operatorname{tg}(\alpha+\omega) - \operatorname{tg}\omega.$$

Through a small conversion and through differentiation, there results

$$f(\alpha) = \frac{\sin\alpha}{\cos\omega\cos(\alpha+\omega)} \qquad f'(\alpha) = \frac{1}{\cos^2(\alpha+\omega)} \qquad f''(\alpha) = \frac{2\operatorname{tg}(\alpha+\omega)}{\cos^2(\alpha+\omega)},$$

where, by taking into account the identity

$$\operatorname{tg}(\alpha+\omega) = \frac{\cos\omega}{\sin\alpha\cos(\alpha+\omega)} - \cot\alpha$$

there is obtained the value

$$\frac{\varphi''(\alpha)}{\varphi'(\alpha)} - \frac{f''(\alpha)}{f'(\alpha)} = 3\cot\alpha - \frac{2\cos\omega}{\sin\alpha\cos(\alpha+\omega)}$$

and finally

$$U = (1+\cos\alpha)\frac{\cos(\alpha+\omega)}{\cos\omega} \qquad V = \frac{3U\cos\alpha}{1+\cos\alpha} - 2$$

Since further

$$\frac{U}{1+\cos\alpha} = \cos\alpha - \sin\alpha \operatorname{tg}\omega$$

α and ω can be determined from the equations

$$\cos\alpha = \frac{V+2}{3U-V-2} \qquad \operatorname{tg}\omega = \cot\alpha - \frac{U}{\sin\alpha(1+\cos\alpha)}$$

The value of U shows that the condition that neither $\alpha + \omega$ nor ω may reach the amount of $\frac{\pi}{2}$ is identical with the condition that U must have a finite positive value. The value of V shows in turn that cos α has the same sign as V + 2 and that we have $3U > V + 2$ even when $V + 2 > 0$. The condition $\cos\alpha > -\lambda^2$ receives, through the value of V, the form

$$V + 2 > -\frac{3U\lambda^2}{1-\lambda^2}$$

and the condition $1 > \cos\alpha > -\lambda^2$ can consequently be written in the form

$$3U > 2V + 4 > -\frac{6U\lambda^2}{1-\lambda^2}$$

The simultaneous condition $U > 0$ is mathematically adequate with finite U but must be carried out for technical reasons.

Since the possibility of achieving eccentric osculation is dependent not only on the functions U and V but also on the function B, there still remains to investigate the latter where special cylinders are to be excluded initially. We then have available the <u>crank mechanism</u> for which

$$B = 1 - \cos\beta - \frac{1}{k}(1-\cos\gamma) \qquad \sin\gamma = k\sin\beta$$

and the coefficient k can be selected arbitrarily. For this, it is best to eliminate k so that the angle γ assumes the role of the latter. Differentiation initially produces

$$B' = \sin\beta - \frac{\sin\gamma\gamma'}{k} \qquad \cos\gamma\gamma' = k\cos\beta,$$

and we obtain from the last equation

$$\gamma' = \operatorname{tg}\gamma \cot\beta \qquad 1-\gamma' = \frac{\sin(\beta-\gamma)}{\sin\beta\cos\gamma},$$

so that there results

$$B' = \frac{\sin(\beta-\gamma)}{\cos\gamma}$$

Differentiation once more produces

$$B'' = \frac{\cos\gamma\cos(\beta-\gamma)(1-\gamma') + \sin(\beta-\gamma)\sin\gamma\,\gamma'}{\cos^2\gamma},$$

and this expression assumes the form

$$B'' = \frac{\sin(\beta-\gamma)}{\sin\beta\cos^2\gamma}\{\cos(\beta-\gamma) + \sin\gamma\operatorname{tg}\gamma\cos\beta\}$$

through insertion of the above values. By utilizing the value derived already above (p. 39)

$$B = \frac{2\sin{}^1\!/_2\,\beta\sin{}^1\!/_2(\beta-\gamma)}{\cos{}^1\!/_2\,\gamma}$$

we obtain

$$\frac{B'}{B} = \frac{\cos{}^1\!/_2\,\gamma\cos{}^1\!/_2(\beta-\gamma)}{\sin{}^1\!/_2\,\beta\cos\gamma} \qquad \frac{B''}{B'} = \frac{\cos(\beta-\gamma)+\sin\gamma\operatorname{tg}\gamma\cos\beta}{\sin\beta\cos\gamma}$$

and then, by utilizing the identity

$$\cos^2{}^1\!/_2\,\gamma = \frac{1+\cos\gamma}{2}$$

the expressions

$$\frac{B'}{B} = \frac{\cot{}^1\!/_2\,\beta}{2}\left(\frac{1}{\cos\gamma}+1\right) + \frac{\operatorname{tg}\gamma}{2} \qquad \frac{B''}{B'} = \operatorname{tg}\gamma + \cot\beta(1+\operatorname{tg}^2\gamma),$$

which are suitable for investigation of the limits of variation of these values. For this purpose, the angle γ must be treated as a variable parameter at a given value of β. Differentiation in regard to the latter produces

$$\frac{\delta}{\delta\gamma}\left(\frac{B'}{B}\right) = \frac{\sin\gamma\cot{}^1\!/_2\,\beta+1}{2\cos^2\gamma} \qquad \frac{\delta}{\delta\gamma}\left(\frac{B''}{B'}\right) = \frac{1+2\cot\beta\operatorname{tg}\gamma}{\cos^2\gamma}.$$

At $\frac{\pi}{2} > \beta > 0$, we have $\cot {}^1/_2\beta > 1$, and it follows from this that not only $\frac{B''}{B'}$ but also $\frac{B'}{B}$ assume infinitely large positive values if γ approaches one of the values $\pm \frac{\pi}{2}$. If the differential quotients are made equal to zero where respectively

$$\sin\gamma = -\operatorname{tg} {}^1/_2\beta \qquad \operatorname{tg}\gamma = -\frac{\operatorname{tg}\beta}{2}$$

we then obtain the respective minima

$$\text{Min.}\frac{B'}{B} = \frac{\cot {}^1/_2\beta}{2}\left(1 + \sqrt{1 - \operatorname{tg}^2 {}^1/_2\beta}\right) \qquad \text{Min.}\frac{B''}{B'} = \cot\beta - \frac{\operatorname{tg}\beta}{4},$$

of which the former is always positive whereas the latter has a negative value of $\beta >$ arc tg 2.

The function

$$X = 2\frac{B'}{B} - \frac{B''}{B'} = \cot {}^1/_2\beta\left(\frac{1}{\cos\gamma} + 1\right) - \cot\beta\left(1 + \operatorname{tg}^2\gamma\right)$$

to be utilized immediately is symmetrical in regard to the value $\gamma = 0$ and assumes an infinitely large negative value when $|\gamma|$ approaches the value $\frac{\pi}{2}$. The differential quotient

$$\frac{\delta X}{\delta\gamma} = \frac{\cot {}^1/_2\beta \operatorname{tg}\gamma}{\cos^2\gamma}\left(\cos\gamma - 2\cot\beta \operatorname{tg} {}^1/_2\beta\right)$$

therefore results in a relative minimum at $\gamma = 0$ and two symmetrical maxima which correspond to the condition

$$\cos\gamma = 1 - \operatorname{tg}^2 {}^1/_2\beta = 2\cot\beta \operatorname{tg} {}^1/_2\beta$$

By taking into account the identities

$$\cot\beta = \frac{\cot {}^1/_2\beta - \operatorname{tg} {}^1/_2\beta}{2} = \cot {}^1/_2\beta - \frac{1}{\sin\beta}$$

there results for the relative minimum X_o

$$X_0 = \cot {}^1/_2\beta + \frac{1}{\sin\beta},$$

and this value is consequently always positive. For the maxima X_m, we obtain initially

$$X_m = \cot {}^1/_2\beta + \frac{1}{\cos\gamma}\left(\cot {}^1/_2\beta - \frac{\cot\beta}{\cos\gamma}\right)$$

and then, by application of the second of the above values of cos γ

$$X_m = \cot {}^1\!/_2 \beta \left(1 + \frac{\operatorname{tg} \beta \cot {}^1\!/_2 \beta}{4}\right).$$

Through the relation

$$\frac{\operatorname{tg} \beta}{4} (\cot^2 {}^1\!/_2 \beta - \operatorname{tg}^2 {}^1\!/_2 \beta) = \frac{1}{\sin \beta}$$

there results the value of the difference

$$X_m - X_0 = \frac{\operatorname{tg} \beta \operatorname{tg}^2 {}^1\!/_2 \beta}{4}.$$

We are now able to investigate the conditions under which an eccentric osculation of the first and/or second order can be accomplished with a standard A-eccentric and B-crank mechanism. For the sake of brevity, we shall utilize here, and in the following examinations, the abbreviated designations

$$\mathfrak{M} = \frac{R'}{K} \qquad \mathfrak{N} = \frac{R''}{R'} \qquad m = \frac{B'}{B} \qquad n = \frac{B''}{B'}$$

with the express assumption $\frac{\pi}{2} > \beta > 0$ so that the relations deduced above receive the form

$$\mathfrak{M} = m\,U \qquad \mathfrak{N} = n + m\,V$$

As far as the <u>eccentric osculation of the first order</u> is concerned, it is immediately clear that $\mathfrak{M} > 0$ represents a condition necessary in all cases since neither U nor m can have a negative value. If the tangential mechanism is utilized, this condition is adequate also mathematically whereas we must require for technical reasons that $\mathfrak{M}$ shall not exceed a certain minimum value dependent on the construction of the machine. If the sinus mechanism is utilized, then we must have

$$\mathfrak{M} \geq m_{\min.} (1 + \cos \omega_m)$$

The same conditions are valid also for machines in which $\omega = 0$. For the sinus mechanism, we here need only make $\cos \omega_m = 1$ and the postulate $2 > U > 0$ resulting for the tangential mechanism can always be complied

with because m can assume an arbitrarily large value. However, if the crank mechanism is replaced by a standard B-eccentric where $m = \cot \frac{1}{2} \beta$, the mathematical conditions for the sinus- and/or tangential mechanism then read

$$\mathfrak{M} - 2 \cot {}^{1}/_{2} \beta > 0 \qquad \text{and/or} \qquad \mathfrak{M} - 2 \cot {}^{1}/_{2} \beta < 0,$$

and it is clear from the latter that even the eccentric osculation of the first order requires the possibility of the application of a finite angle ω if a standard eccentric is utilized in the B-mechanism, in order to be successful in the border cases where the above differences are small.

For <u>eccentric osculation of the second order</u> there is valid, under application of the sinus mechanism,

$$2V > U(U-1) > V(1-\lambda^2) > 0$$

i.e.,

$$2m(\mathfrak{N}-n) > \mathfrak{M}(\mathfrak{M}-m) > m(\mathfrak{N}-n)(1-\lambda^2) > 0.$$

Necessary conditions are therefore

$$\mathfrak{M} > m_{\text{min.}} \qquad \mathfrak{N} > n_{\text{min.}},$$

but compliance with the latter is not adequate since m and n are dependent on each other, even when we disregard that the technical conditions must be carried somewhat further. The respective equation resulting through elimination of cos γ from the formulas deduced above would hardly lend itself, however, to a representation of the necessary and adequate conditions.

The condition

$$3U > 2V + 4 > -\frac{6U\lambda^2}{1-\lambda^2}$$

valid for the tangential mechanism is written as

$$3\mathfrak{M} > 2(\mathfrak{N} - n + 2m) > -\frac{6\mathfrak{M}\lambda^2}{1-\lambda^2}.$$

Of these two inequalities, the first can always be satisfied because 2m - n can receive any arbitrary value located between the positive value X_m and $-\infty$ derived above. The necessary and adequate mathematical conditions are therefore

$$\mathfrak{M} > 0 \qquad \mathfrak{N} > -X_m - \frac{3\mathfrak{M}\lambda^2}{1-\lambda^2}.$$

In order to achieve eccentric osculation of the first and/or the second order in both cases in which the first and/or both conditions are not fulfilled, compound machines or special cylinders must be utilized. In the investigation, we shall consider first the mechanically advantageous application of these cylinders in the B-mechanism. The following method is applicable both to duplex cylinders as well as for general cylinders of the second order. Let the function B be written in the general form

$$B = E + k\varphi(\gamma) \qquad f(\gamma) = c_1 F$$

in which E and F are functions of β. Since B must be symmetrical around zero value, this must therefore necessarily be the case also with E and either with $\varphi(\gamma)$ or with F. Through differentiation and elimination of k and c_1 in the same manner as above (p. 100), there results

$$\frac{B'-E'}{B-E} = \frac{F'}{F} U_1 \qquad \frac{B''-E''}{B'-E'} = \frac{F''}{F'} + \frac{F'}{F} V_1,$$

in which U_1 and V_1 represent the same function of γ as U and V of α. In order to make out of these two equations, containing the three magnitudes B, B', B", one equation in which only m and n occur, let us first subtract $\frac{E'}{E}$ and/or $\frac{E''}{E'}$:

$$\frac{B'E-BE'}{E(B-E)} = \frac{F'}{F} U_1 - \frac{E'}{E} \qquad \frac{B''E'-B'E''}{E'(B'-E')} = \frac{F'}{F} V_1 + \frac{F''}{F'} - \frac{E''}{E'},$$

whereupon the last of these equations is multiplied by the first and divided by the third. The equation so obtained is to be written in the form

$$\frac{F}{F'} \cdot \frac{m\left(n - \frac{E''}{E'}\right)}{m - \frac{E'}{E}} = \Gamma$$

where the function Γ thus introduced is defined by the equation

$$\Gamma = \frac{U_1\left(V_1 + \frac{F}{F'}\left\{\frac{F''}{F'} - \frac{E''}{E'}\right\}\right)}{U_1 - \frac{E'F}{EF'}}$$

The solution of the problem of eccentric osculation of the second order is essentially dependent, in those cases where the simple duplex machine is inadequate, on the properties of this function which are different for the different types of special cylinders and/or compound machines.

When an eccentric is concerned in which the fundamental curve represents a conical section or the parallel curve of one, we have, as demonstrated above (p. 49),

$$B = 1 - \cos\beta + k(1-u) \qquad u^2 = 1 - (q+1)\sin^2\beta.$$

At $q + 1 > 0$, this equation can be written in the general form

$$B = E + k\varphi(\gamma) \qquad f(\gamma) = c_1 F$$

where

$$E = 1 - \cos\beta \qquad \varphi(\gamma) = 1 - \cos\gamma \qquad k = -\frac{\rho}{\rho + qr_0}$$

$$F = \sin\beta \qquad f(\gamma) = \sin\gamma \qquad c_1 = e$$

and, at $q + 1 < 0$, we need insert here only the hyperbolic functions $\mathfrak{Cos}\,\gamma\ \mathfrak{Sin}\,\gamma$ and/or $\sqrt{-(q+1)}$ in place of cos γ sin γ e. This produces initially

$$\frac{F'}{F} = \frac{E''}{E'} = \cot\beta \qquad \frac{E'}{E} = \cot{}^1/_2\,\beta$$

$$\frac{E'F}{EF'} = 1 + \frac{1}{\cos\beta} \qquad \frac{F}{F'}\left(\frac{F''}{F'} - \frac{E''}{E'}\right) = -1 - \mathrm{tg}^2\beta = -\frac{1}{\cos^2\beta}.$$

At $q + 1 > 0$, the functions U_1, V_1 have the values which were determined above for U V with the sinus mechanism, and in which we set $\omega = 0$:

$$U_1 = 1 + \frac{1}{\cos\gamma} \qquad V_1 = \frac{U_1(U_1-1)}{1+\cos\gamma} = \frac{1}{\cos^2\gamma}.$$

At $q + 1 < 0$, the differentiation of the hyperbolic functions produces the

same values where we need to replace cos γ by $\mathfrak{Cos}\,\gamma$. This produces

$$\frac{m \operatorname{tg}\beta\,(n-\cot\beta)}{m-\cot{}^1\!/_2\,\beta} = \Gamma = \left(\frac{1}{u}+1\right)\left(\frac{1}{u}+\frac{1}{\cos\beta}\right),$$

in which u = cos γ and/or $u = \mathfrak{Cos}\,\gamma$, depending on whether $u \lesseqgtr 1$.

It follows from the deduction (p. 49) that u must have a positive value. Otherwise N and ρ would have different signs which would correspond in the hyperbola to transition from one branch to the other and in the ellipse to an angle $\beta > \frac{\pi}{2}$. When $\gamma = 0$, $c_1 = 0$ and the fundamental curve represents a circle if the conditions m = cot ½ β and n = cot β are satisfied. If this is not the case, then k assumes an infinitely large value and the infinitely distant parallel curve of the fundamental curve of the cylinder is a conical section. Since such cylinders cannot be ground with the machine described, this case is mechanically, although not mathematically, excluded. Since we must have $u \neq 1$, this results in the conditions

$$\Gamma > \frac{1}{\cos\beta} \qquad \Gamma \neq 2 \operatorname{tg}\beta \cot{}^1\!/_2\,\beta,$$

in which the latter has meaning only if the value of Γ obtained by m and n does not assume the form $\frac{0}{0}$, which is necessary and adequate in this form.

For eccentric osculation of the first order, we need only solve the equation

$$\frac{\operatorname{tg}\beta\,(mB-\sin\beta)}{B-\sin\beta\,\operatorname{tg}{}^1\!/_2\,\beta} = U,$$

depending on whether U_1 or B has been selected arbitrarily. If this is the case with U_1, then B assumes a real value at any desired value of m. The eccentric osculation of the first order can therefore always be accomplished at an arbitrary value of $\mathfrak{M}$ regardless of whether the sinus or the tangential mechanism is utilized. We can here make $\omega = 0$ and select α arbitrarily after which C is determined by K. If E is then also selected arbitrarily which determines c_1 and γ, then there results from the equation

above B, from which the values of c and k are obtained through which are determined in turn the magnitudes $\rho\, r_o$ characterizing the cylinder in addition to e.

When $\mathfrak{M} = 0$ and consequently when m = 0, we find from the equation for Γ that the product m n must have a finite value and therefore must be $\mathfrak{N} = \infty$ which does not indicate any singularity, however, but merely means that R" has a finite value in the machine curve. The consequence of K = 0 at $R' \neq 0$ would be C = 0 and is consequently impossible to realize by a special cylinder in the B-mechanism. The case $\mathfrak{M} = \infty$ is also mechanically excluded as a finite value of K and of cos $(\alpha + \omega)$ and/or cos ω in the sinus and/or tangential mechanism because, with cos γ = 0, the point of contact of the fundamental curve of the cylinder with the plane would lie on the asymptote and the eccentric would consequently have to be infinitely large. Concerning osculation of the first order, there is consequently applicable the only condition that $\mathfrak{M}$ may not have an infinitely large value. To this should be added merely that the case cos $(\alpha + \omega)$ = 0 can be realized with the sinus mechanism and that therefore this condition is strictly valid only in regard to the osculation of the first order obtainable through the special cylinder.

In examining the possibility of achieving eccentric osculation of the second degree, we have for the sinus mechanism

$$m = \frac{\mathfrak{M}}{U} \qquad n = \mathfrak{N} - \frac{\mathfrak{M}(U-1)}{1+\cos\alpha}$$

and therefore obtain

$$\Gamma = \frac{\mathfrak{M}\ \mathrm{tg}\ \beta \left(\mathfrak{N} - \cot\beta - \frac{\mathfrak{M}(U-1)}{1+\cos\alpha}\right)}{\mathfrak{M} - U\cot\tfrac{1}{2}\beta}.$$

It will be seen immediately that a large positive value of Γ can always be obtained provided $\mathfrak{M}$ is large enough in order to make the numerator

equal to zero by a mechanically appropriate value of U. However, if this is not the case, then osculation of the second order cannot always be achieved. If $U\Gamma$ is designated with [?] and/or x y, then the equation for Γ represents a right-sided hyperbola in which the asymptotes are parallel to the coordinate axes but of which only a part is available because x cannot assume arbitrary values. If then the numerator cannot be made equal to zero, then this is the case only with one part of one branch and y reaches the highest value either at $x = 1 + \cos \omega_m$ or at $x = +\infty$. If the corresponding values of y are designated by Γ_1 and/or Γ_∞ and examination is restricted to the mathematical possibility, i.e., only $U > 1$ is required, we then have

$$\Gamma_1 = \frac{\mathfrak{M} \operatorname{tg} \beta (\mathfrak{N} - \cot \beta)}{\mathfrak{M} - \cot \tfrac{1}{2} \beta} \qquad \Gamma_\infty = \frac{\mathfrak{M}^2 \operatorname{tg} \beta \operatorname{tg} \tfrac{1}{2} \beta}{1 + \cos \alpha},$$

and the mathematical condition states that, when $\mathfrak{M} \gtrless \cot \tfrac{1}{2} \beta$, either Γ_1 or Γ_∞ must be greater than $\frac{1}{\cos \beta}$.

$$n = \mathfrak{N} - \frac{3 \mathfrak{M} \cos \alpha}{1 + \cos \alpha} + \frac{2}{U}$$

is valid for the tangential mechanism and consequently

$$\Gamma = \frac{\mathfrak{M} \operatorname{tg} \beta \left(\mathfrak{N} - \cot \beta - \frac{3 \mathfrak{M} \cos \alpha}{1 + \cos \alpha} + \frac{2}{U} \right)}{\mathfrak{M} - U \cot \tfrac{1}{2} \beta}.$$

Since U can here assume arbitrarily small positive values and since Γ, if U decreases toward zero, approaches the value $\frac{2 \operatorname{tg} \beta}{U}$, osculation of the second order is always possible in the general case. The special case $\mathfrak{M} = 0$, however, requires the product $\mathfrak{M}\mathfrak{N}$ and consequently also $\frac{R''}{K}$ to have a finite negative value which is true moreover for the sinus mechanism but is contained in the condition formulated for that case. In this special case, the normal of the machine curve passes through the center of curvature of the vertex which presupposes a cusp in the evolute and the condition

means that the radius vector has a maximum and/or minimum, depending on whether the latter is larger or smaller than the radius of curvature of the vertex. A different behavior would indicate that a second point R' = 0 existed between the given point and the vertex and/or coincided with the first point at R' = R'' = 0 but such a complicated singularity is therefore completely excluded, except when K = 0.

After we have thus obtained a mechanically applicable value of Γ through appropriate selection of U and α in this manner, a positive value for u results from the quadratic equation

$$\Gamma = \left(\frac{1}{u} + 1\right)\left(\frac{1}{u} + \frac{1}{\cos\beta}\right),$$

after which the other machine constants and the magnitudes determining the cylinder can be obtained in the same manner as above for osculation of the first order. Difficulty results only when $e^2 = 1$ results where the fundamental curve of the cylinder represents a parabola or the parallel curve of one. However, it is apparent from the equation

$$l = \frac{\rho}{2}\left(\frac{1}{\cos\beta} - \cos\beta\right) - r_0(1 - \cos\beta)$$

deduced above (p. 49) that we can set

$$B = 1 - \cos\beta + k \sin\beta \operatorname{tg}\beta$$

By differentiation and elimination of k, we obtain the equation

$$\frac{m B - \sin\beta}{B - \sin\beta \operatorname{tg}{}^1\!/_2\,\beta} = \frac{1 + \cos^2\beta}{\sin\beta \cos\beta},$$

which furnishes the value of B and the other machine constants subsequently result in the customary manner. If we determine the function Γ after differentiating once more in the manner indicated above, then the latter must receive the same value as when we set u = cos β in the general expression which is actually the case.

When utilizing duplex cylinders as eccentrics in the B-mechanism, the simplest circumstances exist if the latter are ground in a standard duplex machine by utilizing a standard B-eccentric. The equation of the machine curve obtained by utilizing such a cylinder is the same as the one just examined with the difference that

$$F = E = 1 - \cos\beta$$

and that f(γ) represents the general sinus or tangential mechanism with the angle of inclination ω_1. There consequently results in general

$$\frac{m \operatorname{tg}^1/_2 \beta\,(n - \cot\beta)}{m - \cot^1/_2 \beta} = \Gamma = \frac{U_1 V_1}{U_1 - 1}$$

and, if the duplex cylinder belongs to the sinus type, i.e., has been ground by utilizing the sinus mechanism,

$$\Gamma = \frac{U_1^2}{1 + \cos\gamma}.$$

We therefore obtain the values $\Gamma\,\Gamma_1\,\Gamma_\infty$ from the analogous values indicated above by multiplying the latter with tg ½ β cot β and the mathematical postulate is $\Gamma > \frac{1}{1 + \cos\gamma}$. With these differences, everything that has been said above on the special cylinders already examined, is valid for these cylinders also, regardless of whether the sinus or the tangential mechanism is utilized with them. If Γ is determined in the customary manner, γ can be selected arbitrarily after which ω_1 can be determined from the resultant value for U_1 in the same manner as ω from U. If we desire to have $\omega_1 = 0$ in those cases where Γ is sufficiently large, we then have

$$\Gamma = \frac{1}{\cos^2\gamma} + \frac{1}{\cos\gamma}.$$

If the duplex cylinder is of the tangential type, there then results

$$\Gamma = \frac{U_1}{U_1 - 1}\left(\frac{3\,U_1 \cos\gamma}{1 + \cos\gamma} - 2\right).$$

The equation

$$3\,U_1^2 \operatorname{tg}^1/_2 \gamma \cot\gamma - U_1(\Gamma + 2) + \Gamma = 0$$

resulting from this has the discriminant

$$(\Gamma + 2)^2 - 12\,\Gamma \operatorname{tg} \tfrac{1}{2}\gamma \cot \gamma,$$

which can be written in the form

$$\left(\Gamma + \frac{2(1-2\cos\gamma)}{1+\cos\gamma}\right)^2 + \frac{12\cos\gamma(2-\cos\gamma)}{(1+\cos\gamma)^2}$$

The roots are therefore real at $\cos\gamma > 0$ and both have positive, with $\Gamma > 0$, and opposed signs at $\Gamma < 0$ so that, at arbitrary values of Γ and $\cos\gamma$, always at least one real positive U_1 is obtained. It follows from this that, with the application of such a cylinder, eccentric osculation of the second order can always be achieved and not even the special case mentioned above is excluded which is equivalent to the fact that the evolute of the machine curve may have two cusps. When $\mathfrak{M} = \infty$, however, we cannot make $\cos\gamma = 0$ which would require an infinitely large eccentric but would have to utilize the sinus mechanism and make $\cos(\alpha + \omega) = 0$. Whether in such a case an osculation of the second order can actually be achieved is not worthwhile of investigation because we will always attempt actually to avoid this which is always possible by selecting a pseudo-duplex curve as machine curve. The duplex cylinder of the tangential type therefore represents the best means in the B-mechanism. Unfortunately, it is inconvenient to make $\omega_1 = 0$ a priori because we must then solve a biquadratic equation in $\cos\gamma$.

When utilizing the <u>special cylinders in the A-mechanism</u>, the investigation can be carried out by the same method. In the equation

$$\frac{R-R_0}{R_0} = C.D \qquad f(\alpha) = c\,\varphi(\beta),$$

in which D is a compound function of α which is represented in the same manner as B above by the equations

$$D = E + k\varphi(\gamma), \qquad f(\gamma) = c_1 F$$

E and F are therefore functions of α and the same postulates must be made on the latter and/or on the function $\varphi(\gamma)$ as above. The derivations of D E F from α are designated by D' ... whereas R' R'' represent the customary derivations from β. If m n then also signify the quotients $\frac{D'}{D}$ and/or $\frac{D''}{D'}$, then the formulas for the function Γ deduced above remain valid unchanged and we need only express m n in $\mathfrak{M}\,\mathfrak{N}\,U$ and α. By differentiation and elimination of C and/or c, we obtain

$$\mathfrak{M} = m\alpha' \qquad \mathfrak{N} = n\alpha' + \frac{\alpha''}{\alpha'}$$

$$\frac{f'(\alpha)}{f(\alpha)}\alpha' = \frac{\varphi'(\beta)}{\varphi(\beta)} \qquad \frac{f''(\alpha)}{f'(\alpha)}\alpha' + \frac{\alpha''}{\alpha'} = \frac{\varphi''(\beta)}{\varphi'(\beta)}.$$

If now the functions U and V are formed in the customary manner from the function $f(\alpha)$ and the derivations of the latter in addition to the function $\varphi(\alpha) = 1 - \cos\alpha$ and the derivations of the latter, there then results

$$\alpha' = U \operatorname{tg} {}^1\!/_2\,\alpha \frac{\varphi'(\beta)}{\varphi(\beta)} \qquad \frac{\alpha''}{\alpha'} = \frac{\varphi''(\beta)}{\varphi'(\beta)} + \frac{\varphi'(\beta)}{\varphi(\beta)}(V - U \operatorname{tg} {}^1\!/_2\,\alpha \cot\alpha),$$

from which results in turn

$$m = \frac{\mathfrak{M} \cot {}^1\!/_2\,\alpha}{m' U} \qquad n = \frac{\cot {}^1\!/_2\,\alpha}{m' U}(\mathfrak{N} - n' - m' V) + \cot\alpha$$

by setting

$$m' = \frac{\varphi'(\beta)}{\varphi(\beta)} \qquad n' = \frac{\varphi''(\beta)}{\varphi'(\beta)}$$

for the sake of brevity. The general expression for Γ then assumes, for $E = 1 - \cos\alpha$, the form

$$\Gamma = \frac{F}{F'} \cdot \frac{\mathfrak{M} \cot {}^1\!/_2\,\alpha}{m' U} \cdot \frac{\mathfrak{N} - n' - m' V}{\mathfrak{M} - m' U}$$

where the value tg α and/or tg ½ α must be applied for $\frac{F}{F'}$, depending on whether the cylinder has been ground in the machine serving for the production of surfaces of the second order or in the duplex machine. In the first case, the postulate $\Gamma > \frac{1}{\cos\alpha}$ exists in the latter if the cylinder of the sinus

type is $\Gamma > \frac{1}{1+\cos\gamma}$, whereas Γ can have an arbitrary value if the latter is of the tangential type. If $\varphi(\beta) = 1 - \cos\beta$, then $m' = \cot^1/_2\beta$ and $n' = \cot\beta$ are valid. We see that in this case also an adequately large positive value of can be obtained with certainty only when, at $\mathfrak{M} > \cot^1/_2\beta$, the numerator can receive an adequately small and either positive or negative value as needed through appropriate selection of U. If this is not the case, success will not be certain when the tangential mechanism is utilized for $f(\alpha)$. In this regard, the utilization of the special cylinder in the A-mechanism is therefore a disadvantage as compared to the utilization in the B-mechanism. However, the first procedure is also always successful if the special cylinder represents a duplex cylinder of the tangential type.

The investigation so far has therefore produced the result that it is necessary to be able to utilize the tangential mechanism in order to achieve in all cases eccentric osculation of the second order. This would be tantamount to saying that the simplest machine will not be adequate in all cases. However, this is still possible, if we take the trouble of grinding two special cylinders, and in two ways.

The simultaneous utilization of special cylinders in the A- and B-mechanisms corresponds to the utilization of a function B in place of $\varphi(\beta)$ where $m' = \frac{B'}{B}$ and $n' = \frac{B''}{B'}$. As was demonstrated above, m' can here have an arbitrary value so that it is always possible to obtain an adequately large positive value for the function Γ of the A-mechanism.

The same purpose can also be realized by special cylinders of a combined type. This designation is intended to comprise the different types of cylinders which can be ground in the duplex machine when utilizing a special cylinder. Of the multiple possible types, we shall mention here only two which will have to be considered in the following. For the sake of simplicity, we shall here assume that all cylinders are ground by utilizing

a plane. As demonstrated above (p. 65), the result is the same when this is not the case but production is somewhat more complicated.

If we write the equation of the machine curve of a duplex cylinder ground with a plane

$$K_1 = a_1 (1 - \cos \alpha_1) \qquad f(\alpha_1) = c_1 (1 - \cos \varphi),$$

and if this cylinder is used as eccentric where a_2 represents the distance of the respective axis from the center of curvature of the vertex, then the displacement in height corresponding to the angle of inclination φ is

$$a_2(1 - \cos \varphi) + K_1.$$

If the cylinder is utilized in the A-mechanism for the grinding of a new cylinder, this produces an A-triplex cylinder and, when this procedure is repeated as many times as desired, an A-multiplex cylinder. We then need each time to replace φ by the angle α applied to the respective grinding process. The machine curve obtained through utilization of a duplex cylinder in an A-mechanism can therefore be represented by the equation

$$K_2 = a_1(1 - \cos \alpha_1) + a_2(1 - \cos \alpha_2) \qquad f(\alpha_1) = c_1(1 - \cos \alpha_2) \qquad f(\alpha_2) = c_2(1 - \cos \beta) \quad *$$

and, for the m-th grinding, the machine curve in general is represented by the m + 1 equation

$$K_m = \sum_{n=1}^{m} a_n(1 - \cos \alpha_n) \qquad f(\alpha_n) = c_n(1 - \cos \alpha_{n+1})$$

where the designation α_{m+1} is applied for β. If now a standard eccentric and/or a duplex, a triplex, etc., eccentric is designated as eccentric of the first, second, etc., order, then the last equations represent the machine curve if an A-multiplex eccentric of the order m is used in the A-mechanism. However, such an eccentric can also be used in the B-mechanism where we shall designate the angle α applied to the production of the eccentric by γ. We now obtain without further difficulty the same m + 1 equations

$$\frac{K}{R_0} = C(1 - \cos \alpha) \qquad E_0 f(\alpha) = \sum_{n=1}^{m} a_n(1 - \cos \gamma_n) \qquad f(\gamma_n) = c_n(1 - \cos \gamma_{n+1}),$$

* Translator's note: Photocopy of original text supplied is illegible, in part.

where a_m represents the distance of the B-axis from the center of curvature of the vertex of the curve of the eccentric and β is designated by γ_m.

It is now evident that, if the coefficients c as well as C in the last method of application are selected arbitrarily, an A-multiplex cylinder of the order m will always offer m machine constants a_n which are obtained through linear equations when as many conditions are prescribed and that consequently, when only the simplest machine is available, an eccentric osculation of the second order can also always be obtained by an A-triplex cylinder, regardless of whether the latter is utilized in the A-mechanism or, with $K \neq 0$, in the B-mechanism. In the former case, the values of $\alpha'_n \alpha''_n$ are determined through differentiation twice of the three last equations after which the first equation represents three linear equations for the desired machine constants a_n, in addition to those obtained from it by twice differentiating, and in the latter case, the procedure is analogous by obtaining $\frac{df(\alpha)}{d\beta}$ and $\frac{d^2 f(\alpha)}{d\beta^2}$ from R'R".

In addition to the A-multiplex cylinders, the <u>B-triplex cylinder</u> is also of benefit in the following calculations. If we write the equation for the grinding of the duplex cylinder as

$$K_1 = a_1(1 - \cos\gamma_1) \qquad E_0 f(\gamma_1) = b_1(1 - \cos\varphi),$$

we then obtain for the second grinding, if the cylinder is utilized in the B-mechanism and b_2 represents the distance of the B-axis from the center of curvature of the vertex of the curve of the eccentric,

$$K_2 = a_2(1 - \cos\gamma_2) \qquad E_0 f(\gamma_2) = b_2(1 - \cos\varphi) + a_1(1 - \cos\gamma_1) \qquad E_0 f(\gamma_1) = b_1(1 - \cos\varphi)$$

and, if the triplex cylinder so produced is utilized in turn in the B-mechanism

$$K_3 = a_3(1 - \cos\gamma_3) \qquad E_0 f(\gamma_3) = b_3(1 - \cos\beta) + a_2(1 - \cos\gamma_2)$$

$$E_0 f(\gamma_2) = b_2(1 - \cos\beta) + a_1(1 - \cos\gamma_1) \qquad E_0 f(\gamma_1) = b_1(1 - \cos\beta),$$

and these equations can also be written in the form

$$\frac{K}{R_0} = C(1 - \cos\alpha) \qquad f(\alpha) = c_3 B_2 \qquad B_2 = 1 - \cos\beta + k_2(1 - \cos\gamma_2)$$

$$f(\gamma_2) = c_2 B_1 \qquad B_1 = 1 - \cos\beta + k_1(1 - \cos\gamma_1) \qquad f(\gamma_1) = c_1(1 - \cos\beta)$$

In the same manner, for the utilization of the B-triplex in the A-mechanism, there results

$$K_3 = a_3(1-\cos\alpha) + a_2(1-\cos\gamma_2) \qquad E_0 f(\gamma_2) = b_2(1-\cos\alpha) + a_1(1-\cos\gamma_1)$$

$$E_0 f(\gamma_1) = b_1(1-\cos\alpha) \qquad f(\alpha) = c_3(1-\cos\beta)$$

and

$$\frac{K}{R_0} = C \cdot D_2 \qquad D_2 = 1-\cos\alpha + k_2(1-\cos\gamma_2) \qquad f(\gamma_2) = c_2 D_1$$

$$D_1 = 1-\cos\alpha + k_1(1-\cos\gamma_1) \qquad f(\gamma_1) = c_1(1-\cos\alpha) \qquad f(\alpha) = c(1-\cos\beta).$$

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It is obvious at this point of the investigation, that it is necessary as a rule to have available a large number of machine constants in order to be able to apply three of them for achieving the prescribed osculation of the second order. However, it should be noted here that the problem can be appreciably simplified in many cases through appropriate selection of the machine curve so that a smaller number of means will be sufficient. It is also possible in other cases to achieve a better osculation of the curve and/or satisfaction of an additional condition by variation of the constants, so-to-speak, in excess. In this regard, we intend to investigate in the near future what means are necessary in the different cases to achieve a centric osculation of the fourth order simultaneously with an eccentric osculation of the second order.

The foregoing will immediately show that this problem may be solved in all cases by an A-quadruplex eccentric even when only the simplest machine is available. If such an eccentric is utilized in the A-mechanism, we then have

$$\mathfrak{A} = \frac{a_4}{R_0}\left(\frac{c_4}{f'_0(\alpha_4)}\right)^2,$$

and the other three constants a_n result by means of three linear equations from the values K R' R''. We shall show by example further below that the solution can also be obtained in certain cases through the selection of an

appropriate parallel curve as machine curve by utilizing only the three constants C c ω of the simplest machine. However, if the machine curve is prescribed, we can generally achieve our objective with simpler means than a quadruplex cylinder, although a special cylinder and/or a compound machine is always necessary in order to obtain the fourth constant.

If we utilize a special cylinder in the B-mechanism, then the equation

$$\frac{K}{R_0} = C\varphi(\alpha) \qquad f(\alpha) = cB,$$

utilized above is valid and we have generally

$$\mathfrak{A} = Cc^2\varphi_0''(\alpha)\left(\frac{B_0''}{f_0'(\alpha)}\right)^2,$$

where all differential quotients are valid for $\beta = 0$. If C c are eliminated in this expression by means of the equations above, there then results

$$B = B_0''\frac{f(\alpha)}{f_0'(\alpha)}\sqrt{\frac{K\varphi_0''(\alpha)}{R_0\mathfrak{A}\varphi(\alpha)}}.$$

In order to make possible further eliminations, B_0'' may not contain any constant characterizing the special cylinder which would be the case if a cylinder of the second order were used. The function B must therefore represent a duplex cylinder so that we obtain, when $E = 1 - \cos\beta$, $B_0'' = 1$. If $\varphi(\alpha) = 1 - \cos\alpha$ and we state, for the sake of brevity

$$\mathfrak{L} = \sqrt{\frac{K}{2R_0\mathfrak{A}}} = \sqrt{\frac{3K}{2R_0^{IV}}}$$

there then results

$$B = \frac{\mathfrak{L}}{\sin{}^1/_2\alpha}\frac{f(\alpha)}{f_0'(\alpha)}.$$

When utilizing the <u>sinus mechanism</u>, as demonstrated above (p. 100), we have

$$f_0'(\alpha) = 1 \qquad f(\alpha) = \frac{1-\cos\alpha}{\sin\alpha}\left(1 + \frac{\cos(\alpha+\omega)}{\cos\omega}\right),$$

from which we obtain

$$B = \frac{\varrho U}{(U-1)\cos {}^1\!/_2\,\alpha}$$

whereas we have, for the tangential mechanism,

$$f_0(\alpha) = \frac{1}{\cos^2 \omega} \qquad f(\alpha) = \frac{\sin \alpha}{\cos \omega \cos (\alpha + \omega)}$$

and obtain

$$B = \frac{4\varrho \cos^3 {}^1\!/_2\,\alpha}{U}$$

m and n are determined in the usual manner so that then also B' and B'' are known functions of U and α and the equations

$$\frac{B' - \sin \beta}{B - \sin \beta \operatorname{tg} {}^1\!/_2\,\beta} = \frac{F'}{F} U_1 \qquad \frac{B'' - \cos \beta}{B' - \sin \beta} = \frac{F''}{F'} + \frac{F'}{F} V_1$$

decide whether the problem can be solved in this manner or not. The circumstances are here the same as in the problem of eccentric osculation of the second order in the standard machine without the utilization of special eccentrics, except for the difference that variation of U and α affords more possibilities. A complete discussion would lead us too far but it is obvious that in those cases where a duplex cylinder does not accomplish its purpose, this is achieved if F represents a duplex cylinder of the tangential type which is expressed by the equations

$$F = E_1 + k_1 \varphi(\gamma_1) \qquad f(\gamma_1) = c_{11} F_1$$

if $f(\gamma_1)$ represents the respective function. The cylinder is then a B-triplex cylinder produced by using a duplex cylinder of the tangential type for the grinding process. By stating

$$\frac{F'}{F} = m' \qquad \frac{F''}{F'} = n'$$

the coefficients can be determined in the manner indicated above through the respective function T_1 where not only U and α but also U_1 and V_1 can be freely selected in the equations above. If $f(\gamma_1)$ represents the tangential

mechanism, then Γ_1 can have an arbitrary value as demonstrated above. However, if only the simplest machine is available so that $f(\gamma_1)$ must represent the sinus mechanism, then there is valid for the equation

$$\frac{m'\left(n' - \frac{E_1''}{E_1'}\right)}{m' - \frac{E_1'}{E_1}} = \frac{F_1'}{F_1} \cdot \Gamma_1$$

the necessary condition

$$\Gamma_1 > \frac{1}{1 + \cos \gamma_1}.$$

Although the values of m' and n' can now be varied through variation of Uα U'V', it is not possible to prove without a very complicated examination that this condition can be satisfied in all cases. However, in cases where this is impossible, we must, in order to still be able to employ the simplest machine, utilize the quadruplex cylinder.

The simultaneous centric osculation of the fourth and of the second order therefore requires at the most a triplex and/or quadruplex cylinder depending on whether the tangential mechanism is available or not and can always be achieved by these means provided that $\mathfrak{L}$ has a finite real value.

If K and R_o^{IV} have different signs, then this must be also the case with $\varphi(\alpha)$ and $\varphi_o''(\alpha)$ as will be seen from the general expression for B above but this then requires a special cylinder in the A-mechanism. This is an expression for the fact that the machine curve, if the A-mechanism consists of a standard eccentric, cannot intersect the circle of curvature of the vertex because the radius vector cannot have a smaller value than the one corresponding to the zero position of the machine. In contrast to this, the case $R_o^{IV} = 0$ can be treated by a special cylinder in the B-mechanism by making $B_o'' = 0$ which produces the equations

$$\frac{K}{R_0} = C\varphi(\alpha) \qquad f(\alpha) = c\varphi(\gamma) \qquad f(\gamma) = c_1\varphi(\beta)$$

If $\varphi(\gamma)$ is designated with B and if B'B" represent the derivations from β, there are then determined only the quotients $m = B'/B$ and $n = B''/B'$ through the eccentric osculation of the second order but since B represents a standard duplex function, U_1 and V_1 are determined in turn through these quotients. It will be evident that

$$m = \frac{df(\alpha)}{d\beta} : f(\alpha) \qquad n = \frac{d^2 f(\alpha)}{d\beta^2} : \frac{df(\alpha)}{d\beta},$$

and the problem therefore is, after calculation beyond the function $f(\alpha)$, exactly the same as the standard problem of osculation of the second order calculated beyond K/R_o. The result therefore is that the same means are adequate in the case $R_o^{IV} = 0$ as in the general case. We need only make $b_3 = 0$ for the respective B-triplex cylinder.

The method of treating the problem is exactly the same if the special cylinder is utilized in the A-mechanism which constitutes the only possibility if K and R_o^{IV} have different signs. Through the value of $\mathfrak{A}$, D is then made into a known function of U and α which is then the case also for D'D" through the values of m n. Only the value of D is different. Through elimination of C and c from the equations

$$\frac{K}{R_0} = C \cdot D \qquad f(\alpha) = c\varphi(\beta) \qquad \mathfrak{A} = Cc^3 D_0'' \left(\frac{\varphi_0''(\beta)}{f'_0(\alpha)}\right)^2$$

there results in general

$$D = \frac{K D_0''}{R_0 \mathfrak{A}} \left(\frac{f(\alpha)\,\varphi_0''(\beta)}{f'_0(\alpha)\,\varphi(\beta)}\right)^2.$$

If the cylinder represents a duplex cylinder and if $\varphi(\beta) = 1 - \cos\beta$, we then have $D_o'' = \varphi_o''(\beta) = 1$ and obtain, through insertion of the respective values of $f(\alpha)$ and $f_o'(\alpha)$, for the sinus mechanism

$$D = \frac{3K}{R_0^{IV}} \cdot \frac{U^2 \operatorname{tg}^2 \tfrac{1}{2}\alpha}{(U-1)^2 \sin^2\beta \operatorname{tg}^2 \tfrac{1}{2}\beta}$$

and/or, for the tangential mechanism,

$$D = \frac{12K}{R_0^{IV}} \cdot \frac{\sin^2\alpha \cos^4 \tfrac{1}{2}\alpha}{U^2 \sin^2\beta \operatorname{tg}^2 \tfrac{1}{2}\beta},$$

Since D may also have a negative value, a point of intersection of the machine curve with its circle of curvature of the vertex located between the point of the eccentric osculation and the vertical point, is possible. At $R_o^{IV} = 0$, we need only state $D_o'' = 0$ which produces the same equations as above where only α and γ change places. Accordingly, we must make $a_3 = 0$ when utilizing the C-triplex cylinder in the A-mechanism. The utilization of a corresponding special cylinder in the A-mechanism therefore makes it possible in all cases to obtain a simultaneous centric osculation of the fourth and of the second order.

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The special case $K = 0$ has been left aside so far in order to be able to discuss it in context. That it requires a compound machine and/or special cylinder has already been stressed. At $R' \neq 0$, there exists a point of intersection of the machine curve with its circle of curvature of the vertex and, in accordance with the foregoing, the A-mechanism must be a compound mechanism and/or contain a special cylinder. D is then equal to zero and the determination of the machine constants is performed in the customary manner. If only an eccentric osculation of the second order is prescribed, the general expression for Γ can be utilized since the latter is valid also at $M = \infty$. It follows from this that a duplex cylinder is always adequate, as in the general case, if it is of the tangential type but that a triplex cylinder may be required in the opposite case. On the other hand, if a simultaneous centric osculation of the fourth order is prescribed, then C is eliminated from the value $\mathfrak{A}$ by means of the equation

$$\frac{R'}{R_0} = CD'\alpha' = CD'U \operatorname{tg} \tfrac{1}{2}\alpha \frac{\varphi'(\beta)}{\varphi(\beta)}$$

which results in

$$D' = \frac{R'}{R_0 A U \operatorname{tg} \frac{1}{2}\alpha \sin^2 \beta \operatorname{tg} \frac{1}{2}\beta} \left(\frac{f(\alpha)}{f'_0(\alpha)}\right)^2$$

for $\varphi(\beta) = 1 - \cos \beta$. The value of n then also makes D'' into a known

function of U and α after which calculation is continued in the customary manner. If R" is also equal to zero, the procedure remains unchanged in spite of this.

When K = R' = 0, the machine curve has a point of contact with its circle of curvature of the vertex and there consequently follows D = D' = 0 for E = 1 - cos α as

$$\cot \tfrac{1}{2}\alpha = \frac{F'}{F} \cdot U_1.$$

It is evident from this that the duplex cylinder must be of the tangential type at F = 1 - cos α and that, if we have only the simplest machine available, eccentric osculation of the second order can be achieved only by a triplex cylinder in the A-mechanism and/or special cylinders in both mechanisms. In the first case and if a B-triplex cylinder is utilized, F constitutes a compound function and, in the latter case, we can make α' = 0 and arbitrarily select D' so that in both C finally results from the value of R". If a simultaneous centric osculation of the fourth order is prescribed, C is eliminated from the value of $\mathfrak{A}$ by means of the equation

$$\frac{R''}{R_0} = CD''\alpha'^2 \qquad \alpha' = U \operatorname{tg} \tfrac{1}{2}\alpha \frac{\varphi'(\beta)}{\varphi(\beta)}$$

after which the determination of the machine constants takes place in the customary manner. The case where the machine curve is to have, in the eccentric point, a contact of the second order with the circle of curvature of the vertex and, in the vertex, a contact of the fourth order with the prescribed curve, can be treated in the same manner. We obtain D = D' = D" = 0 and select C and c so that the product C c^2 receives the prescribed value.

Under certain conditions, the case K = R' = 0 permits the utilization of the special cylinder in the B-mechanism whereas the A-mechanism consists of a standard eccentric. Since the machine curve is here located in its

entirety on one and the same side of this circle of curvature of the vertex, it will be apparent that the centric osculation of the fourth order can be obtained simultaneously with an eccentric osculation of the second order only if the radius vector has a minimum, at a positive value of R_o^{IV}, and a maximum, at a negative value of the latter, in the point of eccentric osculation which is equivalent to the fact that R'' must have the same sign as R_o^{IV} or must also be equal to zero. We obtain $\alpha = 0$ and consequently also $B = 0$ from which results k after γ has been selected arbitrarily. If we then make $\omega_1 = 0$, B' is also known and we obtain the value of α', after c has been selected arbitrarily, after which C results from the equation

$$\frac{R''}{R_0} = C\alpha'^2$$

so that the eccentric osculation of the second order is always possible. If the centric osculation of the fourth order is also prescribed, C is then eliminated from the value of $\mathfrak{A}$ by means of this last equation and the equation

$$\alpha' = \frac{cB'}{f'(\alpha)}$$

which results, by taking into account that $f'(\alpha) = f_o'(\alpha)$,

$$B' = \sqrt{\frac{3R''}{R_0^{IV}}}$$

so that the problem can always be solved, provided that the condition just stated is satisfied.

The case of $R_o^{IV} = 0$ at $K = 0$ also offers no difficulties since we only need to utilize duplex cylinders both in the A- and in the B-mechanisms. In the equations

$$\frac{K}{R_0} = C.D \qquad f(\alpha) = c.B$$

there then must be

$$B = \varphi(\gamma_1) \qquad f(\gamma_1) = c_{11}\varphi(\beta)$$

so that $B_o'' = 0$ and consequently also $\mathfrak{A} = 0$. From the values for R'' and R', there results in the customary manner the equation for the function Γ

inherent in the A-eccentric so that we need only set D = 0 in further calculation.

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It appears from the above that it is always possible to employ four machine constants in such manner that an eccentric osculation of the second order and simultaneously a centric osculation of the fourth order is obtained. The question is now under what conditions the same number of machine constants can also be made available in another manner and this confronts us initially with the problem to obtain an eccentric osculation of the first order in two different points or, abbreviated, <u>two eccentric osculations of the first order</u>. That this can be solved directly in certain cases without the utilization of a quadruplex cylinder is demonstrated in the following manner. When utilizing a duplex cylinder in the B-mechanism, let the machine curve be represented by the equations

$$\frac{K}{R_0} = C\varphi(\alpha) \qquad f(\alpha) = cB$$

where

$$B = F + k\varphi(\gamma) \qquad f(\gamma) = c_1 F \qquad F = 1 - \cos\beta$$

If there are now given two points located on the machine curve which we shall assume to be defined by the values F_1, K_1, R'_1 and/or F_2, K_2, R'_2, we are then required to determine four machine constants through elimination from the equations obtained through these values. In connection with the method utilized above, these are offered by the coefficients c, k, c_1 and the angle ω_1 contained in the function $f(\gamma)$. We therefore arbitrarily select not only the constants $C\omega$ but also the functions $\varphi(\alpha)$ and $f(\alpha)$ by taking into account the conditions valid for the latter which makes the four products cB_1, cB'_1, cB_2, cB'_2 known. Through elimination of k, we obtain in the customary manner, by application of the functions U_1 U_2

formed from the functions $\varphi(\gamma)$ $f(\gamma)$,

$$\frac{B'_1 - F'_1}{B_1 - F_1} = \frac{F'_1}{F_1} \cdot U_1 \qquad \frac{B'_2 - F'_2}{B_2 - F_2} = \frac{F'_2}{F_2} \cdot U_2 \qquad \frac{B_1 - F_1}{B_2 - F_2} = \frac{\varphi(\gamma_1)}{\varphi(\gamma_2)}.$$

From these equations is subtracted on both sides $\frac{F'_1}{F_1}$ and/or $\frac{F'_2}{F_2}$ and/or $\frac{F_1}{F_2}$;

$$\frac{B'_1 F_1 - B_1 F'_1}{B_1 - F_1} = F'_1(U_1 - 1) \qquad \frac{B'_2 F_2 - B_2 F'_2}{B_2 - F_2} = F'_2(U_2 - 1)$$

$$\frac{B_1 F_2 - B_2 F_1}{B_2 - F_2} = \frac{F_2 \varphi(\gamma_1) - F_1 \varphi(\gamma_2)}{\varphi(\gamma_2)}.$$

The fourth of these six equations is multiplied with the third and divided by the sixth whereas the fifth is divided directly by the sixth. We obtain in this manner

$$\frac{B'_1 F_1 - B_1 F'_1}{B_1 F_2 - B_2 F_1} = \frac{F'_1 \varphi(\gamma_1)(U_1 - 1)}{F_2 \varphi(\gamma_1) - F_1 \varphi(\gamma_2)} \qquad \frac{B'_2 F_2 - B_2 F'_2}{B_1 F_2 - B_2 F_1} = \frac{F'_2 \varphi(\gamma_2)(U_2 - 1)}{F_2 \varphi(\gamma_1) - F_1 \varphi(\gamma_2)}$$

and also, through elimination of c_1,

$$\frac{F_1}{F_2} = \frac{f(\gamma_1)}{f(\gamma_2)}.$$

The magnitudes on the left side in these three equations are known whereas the three unknown angles $\gamma_1 \gamma_2 \omega_1$ occur on the right side. If $f(\gamma)$ represents the sinus mechanism, all equations are linear in tg ω_1 when

$$\frac{1}{U_n - 1} = \cos \gamma_n - \sin \gamma_n \operatorname{tg} \omega_1 \qquad f(\gamma_n) = \sin \gamma_n - \operatorname{tg} \omega_1 (1 - \cos \gamma_n) \qquad \varphi(\gamma_n) = 1 - \cos \gamma_n$$

The elimination of this magnitude produces, by utilizing the abbreviated designation

$$A_1 = \frac{B'_1 F_1 - B_1 F'_1}{B_1 F_2 - B_2 F_1} \qquad A_2 = \frac{B'_2 F_2 - B_2 F'_2}{B_2 F_1 - B_1 F_2}$$

the two equations

$$A_1 F_1 \cos(\gamma_1 - \gamma_2) = \cos \gamma_1 (A_1 F_1 - A_1 F_2 - F'_1) + A_1 F_2 + F'_1$$

$$A_2 F_2 \cos(\gamma_1 - \gamma_2) = \cos \gamma_2 (A_2 F_2 - A_2 F_1 - F'_2) + A_2 F_1 + F'_2,$$

which lead to a bi-quadratic equation in cos γ_1 or cos γ_2. If the latter furnishes a real and mechanically applicable value of the angle, we obtain the respective machine constants from this value through linear equations. The variation of the values of C and ω furnish the means for influencing

the bi-quadratic equation. Evidently, the method is not easy and not generally applicable but will be in appropriate cases of benefit if a mechanical simplification is to be achieved at the expense of increased calculations. If the latter does not lead to the objective, it is only necessary to grind the respective quadruplex cylinder but we can also employ the following method.

The machine curve generated through an A-triplex machine with two evolvent eccentrics of the first order is represented by the equations

$$\frac{R-R_0}{R_0} = C_0(1-\cos\alpha) + C_1(\alpha-\sin\alpha) + C_2(1-\cos\gamma) + C_3(\gamma-\sin\gamma)$$

$$f(\gamma) = c_1(1-\cos\alpha) \qquad f(\alpha) = c(1-\cos\beta)$$

if standard eccentrics are utilized otherwise. If here the coefficients cc_1 and the angles $\omega\omega_1$ contained in the function $f(\alpha)$ $f(\gamma)$ are selected arbitrarily, then $\alpha\alpha'\alpha''$ $\gamma\gamma'\gamma''$ are known at a given value of β and we obtain, exactly as with a quadruplex cylinder, four machine constants, i.e., C_n, which result from any arbitrarily formulated prescriptions out of four linear equations. The curve can then be made to pass through four given points and thus achieve four osculations of the order zero or else combine a centric osculation of the fourth order with an eccentric osculation of the first order or another osculation of the order zero, etc. If mechanically unsuitable values should result from the solution of the four linear equations, we then have adequate means for influencing the latter through the variation of the four arbitrarily selected constants. Whether this method or that of the quadruplex cylinder is preferable, will be decided probably only by practical experience. Serious mechanical difficulties do not seem to exist, in any event, with the A-triplex machine, especially not when a B-carriage exists so that the A-axis can be arranged parallel to the B-axis.

It follows from the foregoing that a pseudo-duplex curve is indispensible only when the radius vector of the machine curve constructed in the center of the curvature of the vertex touches the machine curve in one point. We must then select another point as the pole of the coordinate system from which a tangent to the utilized section of the machine curve cannot be drawn. The treatment of the equations

$$\frac{R-R_0}{R_0} = C_0\varphi(\alpha) + C_1\psi(\beta) \qquad f(\alpha) = c\varphi(\beta)$$

is the same as above, except that it is possible to influence the equations through appropriate selection of the pole of the coordinate system. This selection determines R_o and C_1 is then obtained from the radius of curvature of the vertex of the machine curve and the form of the function $\psi(\beta)$. If ρ represents this radius, there are then valid the relations

$$\frac{1}{\rho} = \frac{R_0 - R_0''}{R_0^2} \qquad \frac{R_0''}{R_0} = C_1\psi_0''(\beta).$$

For the sake of mechanical simplification, it is preferable to make $\psi(\beta) = 1 - \cos\beta$ so that $\psi_0''(\beta) = 1$ and the first equation can be written in the form

$$\frac{R-R_0}{R_0} - \left(1 - \frac{R_0}{\rho}\right)(1-\cos\beta) = C_0\varphi(\alpha)$$

from which results in the most simple manner the influence of the variation of R_o. Since the left side of this equation corresponds to the value $\frac{K}{R_o}$ in the customary method, it will be apparent also that the special case corresponding to the case $K = 0$ can always be avoided from which it follows an eccentric osculation of the second order can always be achieved by employing a special cylinder in the B-mechanism.

In order to clearly represent the influence of the variation of R_o on the possibility of obtaining a simultaneous eccentric osculation of the fourth order, it is necessary initially to derive the expression for the flattening value Φ at $R_o'' \neq 0$. Differentiation twice of the generally

valid equation

$$\frac{R^3}{\rho} = \cos^3(\beta - \varphi)(R^2 + 2R'^2 - RR'')$$

initially produces

$$R^3 d^2\frac{1}{\rho} + \frac{3R^2}{\rho} d^2R = 2Rd^2R + 4dR'^2 - Rd^2R'' - R''d^2R - 3(R^2 - RR'')(d\beta - d\varphi)^2.$$

for $\beta = \varphi = 0$. If β represents the independent variable and d σ the element of arc of the machine curve, then

$$d^2\frac{1}{\rho} = \Phi d\sigma^2, \qquad d\sigma = \rho d\varphi = Rd\beta,$$

and we obtain

$$R_0^3\Phi = -\left(\frac{R_0^{IV}}{R_0} + C_1\right) + 3C_1^2(1 + C_1).$$

by applying the relations above. Since further

$$\frac{R_0^{IV}}{R_0} + C_1 = C_0\frac{d^4\varphi(\alpha)}{d\beta^4}$$

there then results

$$C_0c^2\varphi_0''(\alpha)\left(\frac{\varphi_0''(\beta)}{f_0'(\alpha)}\right)^2 = C_1^2(1 + C_1) - \frac{R_0^3\Phi}{3}.$$

The right side of this equation is a polynomial of the third degree in R_o and shall be designated as $F_1(R_o)$. By taking into account that the value of R inherent in the point of eccentric osculation is also a function of R_o, the first equation of the machine curve can be written in the form

$$F_2(R_0) = C_0\varphi(\alpha)$$

Through elimination of C_o, we obtain the condition for employing a standard eccentric in the A-mechanism:

$$\frac{F_1(R_0)}{F_2(R_0)} > 0.$$

Since there is always at least one real value of R_o which makes $F_1(R_o) = 0$ at $F_1'(R_o) \neq 0$ and since a real value of $F_2(R_o)$ corresponds to any value of R_o, this condition can always be satisfied in any case, provided $F_2(R_o)$ does not pass through zero simultaneously with $F_1(R_o)$ and in the opposite direction. In those cases where the standard machine requires a special cylinder in the A-mechanism for obtaining a simultaneous centric and eccentric osculation of the fourth and/or second order, we can therefore, if this method

produces mechanically applicable values, utilize the cylinder in the B-mechanism by grinding the surface with a compound machine.

In non-focal surfaces also, the method of the determination of machine constants remains unchanged for the various problematics, regardless of whether they are ground by employing a horizontal straight-line guidance or with a pseudo-duplex curve as machine curve. In the latter case, when the machine curve is less different from a straight-line than from the corresponding Pascal convolute ["Schnecke"], it may be of advantage to set $\psi(\beta) = \frac{1}{\cos \beta} - 1$ so that we obtain

$$C_0 \frac{d^4\varphi(\alpha)}{d\beta^4} = -R_0^3\Phi$$

since $C_1 = 1$ and $\psi_0^{IV}(\beta) = 5$. We need only to remember here that the form of the function $\varphi(\beta)$ is also influenced because of the construction of the compound machine.

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The principal results in regard to the means required for achieving eccentric osculation can be summarized as follows.

Eccentric osculation of the first order requires, at a finite value of K and a finite positive value of $\mathfrak{M}$, only a standard machine without special cylinder. However, if the tangential mechanism is not available, $\mathfrak{M}$ may not be smaller than a certain value depending on β in that case. In all other cases, a cylinder of the second degree or a duplex cylinder in the A- or in the B-mechanism is adequate. Only in the case of $K = 0$ at $R' \neq 0$, is it necessary to employ such a cylinder in the A-mechanism.

At a finite value of K and a finite positive value of $\mathfrak{M}$, eccentric osculation of the second order can also be achieved with the standard machine without special cylinder, provided $\mathfrak{M}$ exceeds a certain negative value depending on β and the tangential mechanism with

variable B-crank mechanism is employed. If the latter is not the case, the conditions are more complicated. In all cases, a duplex cylinder of the tangential type is adequate. Only when the machine curve intersects its circle of curvature of the vertex in the point of osculation must the latter be employed in the A-mechanism. Utilization of a cylinder of the second degree or of a duplex cylinder of the sinus type in the B-mechanism also achieves the objective, except in certain special cases, provided grinding is affected with the tangential mechanism. If no other than the simplest machine without tangential mechanism is available, an A-triplex cylinder is adequate in every case.

The simultaneous centric and eccentric osculation of the fourth and/or second order requires a special cylinder, when the machine curve is prescribed, and can be achieved in all cases either by means of duplex cylinders in both mechanisms or by means of a B-triplex cylinder if the latter has been ground by employing a duplex cylinder of the tangential type. If no other than the simplest machine is available, an A-quadruplex cylinder is adequate in each case.

Two simultaneous eccentric osculations of the first order can be obtained in certain cases by means of a duplex cylinder in the B-mechanism. In all other cases, an A-quadruplex eccentric or an A-triplex machine with two evolvent eccentrics of the first order is adequate.

By employing the corresponding triplex machine, the order number of an eccentric can be reduced by one unity.

Any desired number m of machine constants are available in linear equations for different problematics if m - 1 and/or m - 2 special cylinders are ground, depending on whether the duplex or the triplex machine is utilized. The A-multiplex cylinder of the order m and/or m - 1 thus obtained can be

employed in the B-mechanism if the machine curve does not intersect its circle of curvature of the vertex which is not the case with the respective evolvent eccentric of the order m - 1 and/or m - 2. In contrast to this, the evolvent eccentric of the order m - 2 can be replaced in the A-triplex machine by two such eccentrics if the sum of the order numbers of the latter remains m - 2.

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IV. Calculation with Non-Spherical Surfaces

In the application of the method for determination of the machine constants demonstrated in the preceding section, difficulties may result because the systems of equations expounded may not furnish sufficiently exact values if these equations are utilized in precisely the form given. Before we enter on a discussion of the methods of mathematical calculation of optical systems with surfaces of the second degree or duplex surfaces, we shall therefore discuss the respective modifications of the equations for some of the simpler cases and indicate by means of a few examples that calculation on the basis of the methods indicated above are not really too complicated.

In the simplest case where it is merely intended to eliminate a Seidel image error, surfaces of the second degree offer such appreciable simplifications, if the system is to be computed mathematically, that they should be selected whenever possible. The Seidel formulas, in the form which I have given them, directly furnish the flattening value Φ of the surface, and for a surface of revolution of the second degree whose meridian curve has the radius of curvature of vertex p and the numerical eccentricity e, there is valid the relation

$$\Phi = -\frac{3e^2}{p^3}.$$

At a positive value of the product p Φ, the meridian curve therefore represents an ellipse whose shorter axis coincides with the axis of revolution. The general equation

$$y^2 = 2px + qx^2$$

furnishes, if the X-axis coincides with the axis of revolution and y therefore represents the independent variable, for x = y = 0:

$$pd^2x = dy^2 \qquad pd^4x + 3q(d^2x)^2 = 0,$$

from which results, because

$$\Phi = \frac{d^4x}{dy^4} - 3\left(\frac{d^2x}{dy^2}\right)^3$$

in the vertex of a curve symmetrical around the X-axis, the expression

$$\Phi = -\frac{3(q+1)}{p^3}$$

which represents, at a negative value of the product $p\Phi$, the relation given above of the flattening value to the eccentricity but represents, in the contrary case, by means of the relation $q = -\frac{B^2}{A^2}$ the relation of the semi-axis B perpendicular on the axis of revolution to the same axis A coinciding with the latter in which $B > A$. From any arbitrary value of Φ, we consequently obtain in this manner the absolute constants of the corresponding conical sections when the radius of curvature of the vertex is prescribed.

In most cases, the present purpose will be achieved by substituting the corresponding surface of the second degree for a convex spherical surface and the former can then be ground directly as such a surface by the method described above. However, in the cases where a concave non-spherical surface is to be preferred for special reasons or where, e.g., a biconcave lens is required, a duplex surface must be utilized. From the mechanical viewpoint, the greatest advantage then lies in selecting a standard surface as simple as possible in accordance with the proposal demonstrated above. However, if the system is to be calculated, it may be, especially in the cases where the diaphram cannot be incorporated in a medium bonded by the non-spherical surface, of great advantage to have the surface osculate a surface of the second degree as accurately as possible. Since we can obtain the intersection of a given ray with such a surface through solving an equation of the second degree whereas the former can be determined only by more complicated calculation for duplex

surfaces, we can initially calculate with the surface of the second degree and subsequently utilize the values obtained either directly or as first approximate values depending on the degree of accuracy required. Under certain circumstances, it may also be advantageous to utilize a duplex surface which has optimum possible osculation with a prescribed convex surface of the second degree.

If we now want to calculate a duplex surface in consequence of these reflections and this surface has a centric osculation of the eighth order with a surface of the second degree, let us first correspondingly differentiate the equation of the second degree of the curve just utilized. At $n > 1$, we have for the vertex:

$$p d^{2n}x + \frac{q}{2} d^{2n}(x^2) = 0,$$

in which

$$d^6(x^2) = 15 d^4x d^2x \qquad d^8(x^2) = 28 d^6x d^2x + 35 (d^4x)^2$$

so that we obtain, after exchange of the variables:

$$y'' = \frac{1}{p} \qquad y^{IV} = -\frac{3q}{p^3} \qquad y^{VI} = \frac{45q^2}{p^5} \qquad y^{VIII} = -\frac{1575q^3}{p^7}.$$

By substitution of these values of the formulas deducted above (p. 72) we obtain the differential quotients in the polar coordinate system $r'' = 0$:

$$r^{IV} = 3pe^2 \qquad r^{VI} = -15pe^2(3e^2 + 1) \qquad r^{VIII} = 63pe^2(25e^4 + 25e^2 + 1)$$

and for a parallel curve (p. 74) if we set

$$\lambda = \frac{R_0 - p}{R_0}$$

$$R^{IV} = r^{IV} \qquad R^{VI} = r^{VI} + 10\lambda \frac{r^{IV^2}}{p} \qquad R^{VIII} = r^{VIII} + 56\lambda \frac{r^{VI} r^{IV}}{p} + 280\lambda^2 \frac{r^{IV^3}}{p^2}.$$

For the magnitudes (p. 77) to be applied to the calculation of the machine constants, we therefore obtain

$$\mathfrak{A} = e^2(1-\lambda) \qquad \mathfrak{B} = -e^2 + 2\lambda e^2$$
$$\mathfrak{C} = 5e^4 + 5e^2 - 8\lambda e^2(3e^2+1) + 24\lambda^2 e^4$$
$$C_s = 3e^2 - 4\lambda e^2(e^2+1) + 4\lambda^2 e^4$$
$$C_t = 2e^4 + 3e^2 - 4\lambda e^2(3e^2+1) + 12\lambda^2 e^4.$$

Since p 0, the parallel curve lies on the convex side of the conical section in the case $1 > \lambda > 0$ whereas, at $\lambda > 1$, the latter is located on the concave side beyond the center of curvature of the vertex and, at $\lambda < 0$, between the latter point and the vertex. The case $\lambda = 1$ corresponds to an infinitely distant parallel curve. Since the base curve represents a conchoid with this curve as basis, we obtain the values for the latter if $\lambda = 1$ is inserted in the values of $\mathfrak{B}\,\mathfrak{C}\; C_s\; C_t$ whereas $R_o = p$ for the base curve and, in concordance herewith, $\lambda = 0$ must be inserted in the value of $\mathfrak{A}$.

For the simplest machine without crank mechanism and without carriage, we have $c^2 = c_s$. From the expression above, we obtain the value $C_s = 3e^2$ and/or $C_s = -e^2$ for $\lambda = 0$ and/or $\lambda = 1$. If we set $c_s = 0$, this produces two real values of λ of which one consequently corresponds to a parallel curve located on the convex side. Since the roots of the quadratic equation in λ both have positive or opposed signs, depending on whether e^2 is or is not positive, the other parallel curve determined by the condition $C_2 = 0$ lies on the concave side and, in the first case, beyond the center of curvature of the vertex and, in the second case, between this point and the surface. If we write the value of C_s as

$$C_s = -e^2 - (e^2-1)^2 + (e^2+1-2\lambda e^2)^2,$$

it is immediately apparent that C_s possesses a minimum with negative value but no maximum which is true also of the cases concerning ellipsoids of rotation with the shorter axis as axis of revolution in which $e^2 < 0$. Furthermore, since $\mathfrak{B} = \operatorname{ctg} \omega$ for this type of machine and c is always

made positive, it follows from the value of ϑ that ω has the same or the opposed sign of e^2, depending on whether $\lambda \gtrless 0.5$ whereas $\omega = 0$ when λ has precisely this value.

For the case $e^2 > 0$, there initially follows from this in regard to convex surfaces that the latter can be ground in general either with a cylinder or with a concave semisphere since C_s is positive both at a very small and at a very large value of λ. The former method will be preferable for mechanical reasons as soon as the cylinder would not have to have too small a diameter which is the case when the ratio of numerical eccentricity to parameter is large. The larger e^2 is, the more closely will the two values of λ which correspond to $C_s = 0$, approach the values 0 and 1. If therefore the utilization of the cylinder at large numerical eccentricity is impossible for mechanical reasons, we can then always grind with the concave semisphere of relatively large radius where, however, a limit for the opening of the ground surface is established by the fact that the latter may not have a radius of curvature larger than that of the cup. The concave surfaces, on the other hand, can always be ground by means of a small sphere. Although this produces a value of c which is as much larger as numerical eccentricity is greater but, since ω has a negative value which permits a larger angle α from the mechanical viewpoint, this counteracts a narrowing of the opening.

At $e^2 < 0$, the convex surface can in general be ground with a plane where $c = \sqrt{-e^2}$ and $\operatorname{tg} \omega = -c$. However, at a large value of $|e^2|$, mechanical difficulties are created by the magnitude of $|\omega|$ which can be eliminated, however, through grinding with a concave spherical surface. Because the value $\operatorname{tg} \omega = -1$ corresponds to the case $\lambda = \infty$, it is possible to obtain, in the cases occurring in practice, a mechanically applicable

value of ω through an adequately large value of λ where, however, the large value of c narrows the opening through the dimension of the angle $\alpha + \omega$. Since a parallel surface is located between the surface and the center of curvature of the vertex of the latter for which $C_s = 0$ for this surface, a numerically adequately large negative value of λ produces a real value of c so that concave surfaces can consequently be ground by the application of an arbitrarily selected value of this machine constant. It should merely be remembered here that too large a value of the positive angle ω is obtained at too small a value of c and that, on the other hand, when c is too large, the opening is narrowed both by the large value of $\alpha + \omega$ as well as through the condition that the non-spherical surface may not have a radius of curvature smaller than that of the abrading surface.

With this review, I have merely intended to show that the calculations in the determination of the machine constants concerning centric osculation of the eighth order are relatively simple. When other surfaces than those of the second degree are concerned, the latter can then obviously not be represented so simply and clearly that the entire category of the surfaces can be treated at once. However, if a given surface is concerned, the discussion above can be carried out in the same manner and the calculations remain the same as soon as the differential quotients have been derived.

That such a derivation in other cases also does not necessarily require endless calculations will now be shown by the example of the Cartesian oval. If s s' designate the distances, considered as positive in the direction of motion of the light, of the points of focus and/or image from the vertex of the curve and nn' the indices of refraction, then the optical length from the point of focus to the point of image is equal to Δns where the designation of the optical invariant

$$\Delta ns = n's' - ns$$

If q q' represent the distances of the same point, considered as positive in the same way, from an arbitrary point of the curve, then the condition that the optical length between the two points shall be the same on a ray of like incidence as on the axis,

$$\Delta nq = \Delta ns,$$

and we obtain, through the relation

$$q^2 = (x - s)^2 + y^2$$

and through the similar relation valid for q', the equation of the curve in Cartesian coordinates. In the successive differentiations, y shall be treated as an independent variable and $x''x^{IV}$ ... and/or $q''q^{IV}$ shall designate the respective derivations of x and/or q. Differentiation two times results in

$$qd^2q = -sd^2x + dy^2$$

for x = y = 0 where q = s, i.e.,

$$s(q'' + x'') = 1.$$

This equation is multiplied with $\frac{n}{s}$ and subtracted from the similar equation valid for the image medium where we obtain

$$x''\Delta n = \Delta\frac{n}{s}$$

by considering that $\Delta nq'' = 0$. Since x'' represents the reciprocal value of the radius of curvature of the vertex r, the above equation furnishes the relation of this radius to the distances s s' and we obtain

$$q'' = \frac{1}{s} - \frac{1}{r}.$$

In the subsequent differentiations, there result for $\nu > 1$ the equations

$$s(d^{2\nu}q + d^{2\nu}x) = (X^{II\nu} - Q^{II\nu})\,dy^{2\nu},$$

where

$$X^{IV} = 3x''^2 \qquad X^{VI} = 15x^{IV}x''$$
$$X^{VIII} = 28x^{VI}x'' + 35x^{IV^2}$$

and where the binomial quotients of the even-number members are applied unchanged but those of the odd-number number members after division by 2, and the magnitudes $Q^{II\nu}$ are formed in the same manner. These equations are treated in the same manner as above, i.e., multiplied initially with $\frac{n}{s}$ and subtracted from the similar equations valid for the image medium. The equation so obtained produces, together with the equation $\Delta_n q^{II\nu} = 0$, the differential quotients $x^{II\nu}$ and $q^{II\nu}$ which also furnishes us the value for $X^{II(\nu+1)}$ and $Q^{II(\nu+1)}$. After these, we immediately obtain the differential quotient of the desired order of the equation of curve $x = f(y)$ where we need only know the respective binomial quotients.

A useful subsequent and small transformation of the values is best illustrated by the example $\nu = 2$. The equation

$$s(q^{IV} + x^{IV}) = 3(x''^2 - q''^2)$$

initially furnishes

$$x^{IV}\Delta n = 3x''^2\Delta\frac{n}{s} - 3\Delta\frac{nq''^2}{s},$$

but receives, by the substitution of $\Delta\frac{n}{s} = x''$ and by taking into account that $n^2q''^2$ is an optic invariant, the form

$$x^{IV} - 3x''^3 = -\frac{3n^2q''^2}{\Delta n}\Delta\frac{1}{ns},$$

from which is obtained

$$q^{IV} = -x^{IV} + \frac{3(x''^2 - q''^2)}{s}$$

In the same way, there further results

$$x^{VI} - 15x^{IV}x''^2 = -\frac{15n^2q^{IV}q''}{\Delta n}\Delta\frac{1}{ns}$$

and

$$q^{VI} = -x^{VI} + \frac{15(x^{IV}x'' - q^{IV}q'')}{s}$$

$$x^{VIII} - 28x^{VI}x''^3 - 35x^{IV^2}x'' = -\frac{28n^2q^{VI}q'' + 35n^2q^{IV^2}}{\Delta n}\Delta\frac{1}{ns}.$$

After this, the calculations in the determination of the machine constants for an eccentric osculation shall be discussed by means of an

example. If a magnitude important for the calculation is obtained as the difference of two approximately equal magnitudes, there is then necessary in general a transformation in order to achieve sufficient accuracy, and in certain cases it is preferable to employ series development for this. Since it would be much too complicated to enter here on a large number of the necessary calculation methods, we shall select an example where such transformations occur at different points. If we are concerned with improving a duplex surface without utilization of a crank mechanism by the introduction of an evolvent eccentric of the first order into the A-mechanism, c and ω can be selected freely and C and k_1 can then be determined through the condition of an eccentric osculation of the first order. The best curve is determined in this manner by variation of c and ω. If c receives here a low value, we must transform, especially if β is low at the same time, both the equations representing the general sinus mechanism as well as those representing the general tangential mechanism. In the former mechanism, we obtain α from the formula

$$\frac{\sin(\alpha+\omega)-\sin\omega}{\cos\omega}=c(1-\cos\beta).$$

Let the magnitude on the right side be designated as h. If we substitute $1-\cos\beta$ by $2\sin^2\frac{\beta}{2}$ or by $\sin\beta\operatorname{tg}\frac{\beta}{2}$, h can then be calculated with any desired accuracy. However, in order to obtain an accurate value for α at small h and when ω is not small at the same time, it is best to use the value resulting from the equation above as a first approximate value α_1 in the equation

$$\sin\alpha_2=h+2\operatorname{tg}\omega\sin^2\frac{\alpha_1}{2}$$

highly suitable for iteration, and thus obtain a better value of α_2 which can be made as accurate as desired by repetition of the procedure.

The smallest values of h do not occur in the calculation of the

machine constants but in the trigonometric pursuit of a ray encountering the non-spherical surface in the proximity of the vertex. In spite of this, they shall be treated in this connection. It will be apparent that the method above is successful but, with a very small h, a sufficiently accurate value will be obtained more quickly with a series. The original equation can be written as a quadratic equation in sin α = x in the form

$$x = h + \frac{x^2 \operatorname{tg}\omega}{2} + \frac{\cot\omega}{2}(x-h)^2$$

The first three differentiations furnish

$$dx = dh$$

$$d^2x = \operatorname{tg}\omega\, dx^2$$

$$d^3x = 3\operatorname{tg}\omega\, dx\, d^2x,$$

and, after the fourth differentiation, we obtain, in consideration of the fact that

$$\operatorname{tg}\omega + \cot\omega = \frac{\cot\omega}{\cos^2\omega}$$

$$d^4x = 4\operatorname{tg}\omega\, dx\, d^3x + \frac{3\cot\omega}{\cos^2\omega}(d^2x)^2$$

$$d^5x = 5\operatorname{tg}\omega\, dx\, d^4x + \frac{10\cot\omega}{\cos^2\omega} d^2x\, d^3x$$

$$d^6x = 6\operatorname{tg}\omega\, dx\, d^5x + \frac{\cot\omega}{\cos^2\omega}[15\, d^2x\, d^4x + 10(d^3x)^2]$$

etc., where we need only apply the binomial coefficients in the manner already indicated. By taking into consideration the numbers up to and including the sixth order, we find the series

$$\sin\alpha = h + \frac{h}{2}\sum_{n=1}^{n=5} h^n \operatorname{tg}^n\omega + \frac{h^3}{8\cos^3\omega}(h\operatorname{tg}\omega + 3h^2\operatorname{tg}^2\omega + 6h^3\operatorname{tg}^3\omega) + \frac{h^6\operatorname{tg}\omega}{\cos^4\omega}.$$

In the tangent mechanism, the equation

$$\operatorname{tg}(\alpha+\omega) - \operatorname{tg}\omega = h,$$

where the tangent of the sum of the angles is expressed by the tangent of the two angles, is brought into the form

$$\operatorname{tg}\alpha = \frac{h\cos^2\omega}{1 + h\sin\omega\cos\omega}$$

which permits calculation of α with any desired accuracy.

The equations

$$C(1-\cos\alpha)+Ck_1(\alpha-\sin\alpha)=\frac{K}{R_0}$$

$$C\sin\alpha+Ck_1(1-\cos\alpha)=\frac{R'}{R_0\alpha'}$$

serving for the calculation of the machine constants Ck_1 produce, for the determinant D which constitutes after solution of the latter the numerator of the values obtained, the expression

$$D=2(1-\cos\alpha)-\alpha\sin\alpha,$$

which cannot be calculated sufficiently accurately in the usual manner when the angle α is small. However, by means of the two series

$$2(1-\cos\alpha)=2\frac{\alpha^2}{2!}-2\frac{\alpha^4}{4!}+2\frac{\alpha^6}{6!}-\cdots$$

$$\alpha\sin\alpha=2\frac{\alpha^2}{2!}-4\frac{\alpha^4}{4!}+6\frac{\alpha^6}{6!}-\cdots$$

there results

$$D=2\frac{\alpha^4}{4!}-4\frac{\alpha^6}{6!}+6\frac{\alpha^8}{8!}-\cdots,$$

which makes it possible to achieve any desired accuracy.

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After this brief explanation of the calculations required for the determination of the machine constants, we shall now turn to the actual methods of theoretical calculation.

In the mathematical examination of the design calculations for an optical instrument, that is, in re-calculation of the latter, the emphasis lies in general on the trigonometric pursuit of individual rays. This is done in connection with a still prevailing concept of the predominant importance of the cross-section of a bundle of rays for the optical projection.

However, since the latter is effected through the fusion of rays which affects in turn in general only the closely adjacent rays, the cross-sections of the caustic surfaces in many cases have so large an importance for the sharpness of the image that the cross-sections of the bundles of rays become of secondary importance. The simplest way of convincing ourselves of this is to project the image of the glowing filament of an electric bulb on a screen by means of a simple bi-convex lens with a large aperture. If we then adjust to the smallest circle of diffusion, then the image does not give any idea of the object whereas the latter can be clearly seen as soon as the distance is adequately enlarged in order to bring the cusp of the evolute on the screen. The large circle of diffusion so created mainly has the effect of a veil. However, if the lens is tilted so that the caustic surface receives a less favorable form, there occurs an appreciable deterioration of the image, although the magnitude of the diffusion figure does not appreciably increase with the most favorable adjustment. These simple experiments teach us without any possibility of error that, for non-homocentric bundles of rays, the cross-section of the caustic surface plays the essential and that of the bundle of rays a minor role. Since the greater part of the optical images further should fall in the last analysis on the retina of the eye or on the photographic plate and since in either one or the other case primarily differences of intensity are decisive, this is a further circumstance contributing to make the veil created by the circle of diffusion relatively without effect.

However, that this conclusion is not immediately applicable to all possible optical images can be seen simply by the fact that a whole category of such images is not offered either to the eye or to the photographic plate. This is especially the case with the images of the pupils and/or

the apertures of optical instruments, regardless of whether we are concerned with projecting an opening entirely within or entirely without another. For example, this last problem is formulated by the conditions of the non-reflecting ophthalmoscopy. In these cases, it is evident that the smallest circle of diffusion maintains its former place of predominance.

However, disregarding these and similar cases, since the extent and form of the caustic surface plays a decisive role, it is obvious that, in the general case, re-calculation is not primarily intended for the trigonometric pursuit of the largest possible number of rays but is capable of giving the most reliable knowledge of the projection through the determination of the properties of the caustic surfaces in the close proximity of a lesser number of trigonometrically pursued rays. This is equivalent to saying that the [Abbe] Laws of Imagery of Higher Order must be employed. This is, of course, done in regard to the laws of the first order in general where we are concerned with the projection of the image of an eccentrically located focal point through an axial-symmetric system by calculating the tangential and sagittal image point on the main ray passing through the center of the diaphram. However, in order to obtain more detailed knowledge of the caustic surface, these image points must be calculated also for other rays starting from the same object point or else we must, by applying the laws of the second order, determine the asymmetrical values of the bundle of rays parallel to the main ray. In any case, it is preferable to calculate the transverse asymmetrical value which, in systems of revolution, does not require the knowledge of the differential quotients of the third order of the equation of the meridian curve of the non-spherical surface. It will therefore be evident that the

trigonometric pursuit of rays which do not intersect the axis, so-called "warped" rays, is practically of no value without the time-consuming effort of the calculation of the image points on the latter and has a higher value with this calculation than the calculation of the transverse asymmetry only then when several warped rays are made the basis. The advantage which is afforded through this extremely difficult effort of calculation would, however, be able to make itself felt only with the very largest apertures from which follows that the trigonometric pursuit of warped rays should be utilized only in the most infrequent cases. We shall therefore indicate the pertinent point, only for the sake of completeness, the method by which the intersection of a given warped ray with a non-spherical surface is obtained.

What has been said here on the application of the laws of higher order is valid not only for the caustic surfaces but -- all things being equal -- also for the other magnitudes determining the image. For example, if the abberration of a given ray is corrected for one axial point and the sinus condition is thus satisfied, we then obtain, by the investigation of the sinus relation parallel to other rays, adequate understanding of the respective relations but must, parallel to these rays, calculate also the tangential image point in addition to the respective coefficient of enlargement as will be explained further by means of the example below. The same is true also for distortion. As far as the image surfaces are concerned, we obtain, by application of the laws of second order, the tangients corresponding to the respective image points but can obviously replace this calculation by the determination of a larger number of image points. As disclosed by this review, an exact representation of the methods of recalculation must also take into consideration the laws of the second order.

In the derivation of the formula, I have retained as far as possible the designations of the distances and angles utilized previously for spherical surfaces and considered the latter positive in accordance with the usual or customary method. The axis of revolution therefore represents the X-axis of the coordinate system whose starting point coincides with the vertex of the surface and the distances on the latter are calculated as positive with a refraction in the direction of the motion of light. The incident and/or refracted ray intersects the axis in a point whose distance from the vertex of the surface is s and/or s' and forms the angle u and/or u' with the axis. The coordinates of the intersection of the two rays with the meridian curve of the surface are xy and the sign of the angles is determined through the relations tg $u = \frac{y}{s-x}$ and $|u| < \frac{\pi}{2}$ as well as the similar ones valid for the image medium. This consequently excludes from consideration the case where the projection of the motion of light taking place on a warped ray on the axis of the former would be opposite to that of the latter. In the point of the curve xy is drawn the normal which forms the angle φ with the axis and has the length N between the point of the curve and the intersection with the axis whereas M represents the distance of the intersection from the vertex. The sign of this last magnitude is therefore given and the sign of N shall be the same. The case M = 0 is consequently excluded from consideration. Through the two equations

$$y = N \sin \varphi \qquad M - x = N \cos \varphi$$

φ is definitely determined so that, e.g., at M (M - X) $<$ 0, a value of $|\varphi| > \frac{\pi}{2}$ is obtained. The sign of the angle of incidence and/or refraction in i or i' is determined by the conditions

$$\frac{\sin i}{\sin u} = \frac{s - M}{N} \qquad |i| < \frac{\pi}{2}$$

as well as through the similar ones valid for the image medium. Due to these observations, the equations $\varphi = u + i = u' + i'$ always produce a value of $|\varphi| < \pi$. The distance of the intersection of the incident and/or refracted ray with the axis from the curve point xy is designated by q and/or q'. Due to the conditions noted in regard to the angles u u', the relations

$$q \sin u = y \qquad q \cos u = s - x$$

as well as the similar ones valid for the image medium signify that q q' have respectively the same sign as s s' and are therefore calculated as positive in the direction of the motion of light. The same is true of the distances p p' of the tangential focal points corresponding to an axial point located in any desired medium from the point xy. The tangential and/or sagittal coefficients of an enlargement in the projection of this axial point in the present object and/or picture medium are $\chi, \chi_{,,} \chi', \chi'_{,,}$. In the same manner, the tangential and/or sagittal focal distances $\tau\, \varsigma\, \tau'\, \varsigma'$ and coefficients of enlargement $K, K_{,,} K', K'_{,,}$ correspond to an extra-axial point located in any desired medium. The indices of refraction are designated by n n'. The sign of the latter is positive or negative depending on whether the light in the respective medium moves or moves not in the direction of the positive X-axis. In a reflection, we therefore must set n' = -n but the distances on the axis are calculated as positive for both media in one and the same direction. In certain cases, it is preferable to make both indices of refraction negative. For example, if we are concerned with a lens reflecting from the back, the positive direction of the X-axis is permitted to coincide with the direction of the motion of light in the object medium, in order to avoid a change of sign during the calculation, and must make then both indices of refraction negative for the

refraction arriving after reflection has taken place. However, for the sake of simplicity, it is preferable in the present representation to employ positive indices of refraction at single refraction which is equivalent to the fact that the distances in the direction of the motion of light are calculated as positive. Furthermore, ρ is to design the radius of curvature in the vertex and $\rho, \rho_{\shortparallel}$ the tangential and/or sagittal radius of curvature in point xy where consequently $\rho_{\shortparallel} = N$. The sign is determined by the fact that a radius of curvature is defined as the distance of the center of curvature from the point at the surface where the positive direction on the normal is determined by the sign of M. Finally, D is to designate the power of refraction in the vertex and $D, D_{\shortparallel}$ the tangential and/or sagittal power of refraction in the point xy.

Since the determination of the point of incidence xy and its respective magnitudes MN$\varphi\rho$ takes place by different methods depending on whether a surface of the second degree or a duplex surface is utilized, whereas the calculations, after these magnitudes are known, are carried out in both cases in one and the same manner, it seems therefore indicated to discuss the determination of these magnitudes only later and to initially assume them here as known. Moreover, we know the indices of refraction and the magnitudes su characterizing the incident ray and, where the application of the laws of higher order is concerned, at least the distance of $q - p$ and the coefficients of enlargement $\chi, \chi_{\shortparallel}$. These correspond to an axial point which is located in the respective medium on the respective ray but may otherwise have a different significance. If we are treating with an optical system which is intended to project an axial point as sharply as possible, they then correspond to this point which therefore constitutes a focal point. However, if the optical system is to be

employed for the projection of extra-axial points, these same magnitudes correspond to the diaphragm center whereas, in regard to the focal point, the distances $q - \tau$ and $q - \zeta$ as well as the coefficients of enlargement K_I, K_{II} are known. If one or the other point is located in the respective first medium, we then have to set $q - p = 0$ and/or $\tau = \zeta$ and $\chi_I = \chi_{II} = 1$ and/or $K_I = K_{II} = 1$. If here one of the points is infinitely distant, it is simplest to set $\left|\frac{\chi_I}{p} = \frac{\chi_{II}}{q} = -1\right.$ and/or $\left|\frac{K_I}{\tau} = \frac{K_{II}}{\zeta} = -1\right.$. The coefficients of enlargement belonging to the second medium then no longer represent figures but have the dimension of a length. However, if an infinitely large value of a focal distance in the respective first medium corresponds to a focal point or diaphram center located at a finite distance, then we always know the magnitude of the form $\frac{\chi}{p}'$. Finally, at infinitely large s, the respective focal distances are determined through values of the form $p + x$.

Since φ and u are known, the equation $\Delta n \sin i = 0$ is sufficient in order to make known all angles. We then obtain s' from the equation

$$\left| s' - M = \frac{N \sin i'}{\sin u'}, \right.$$

which furnishes, however, an inaccurate value if s' is very small in relation to M. In those cases, the equations

$$\left| q = \frac{N \sin \varphi}{\sin u} \qquad \Delta q \sin u = 0 \qquad \Delta s = \Delta q \cos u \right.$$

furnish a sufficiently exact value. Both the former are also used for the determination of q q'; after which the last can be employed as control. If qq' are eliminated from the latter, it can be written in the form

$$\left| \Delta s = \frac{N \sin \varphi \sin \Delta i}{\sin u \sin u'} \right.$$

which is more convenient in those cases where we are concerned exclusively with the trigonometric pursuit of a ray. The sagittal coefficient of enlargement corresponding to the axial point and the sagittal power or

refraction produce the formulas

$$\Delta n \chi_{II} \sin u = 0 \qquad D_{II} = \Delta \frac{n}{q} = \frac{\Delta n \cos i}{N} = -\frac{n n' \sin \Delta i}{I N},$$

where, in the last expression very appropriate for trigonometric calculation, I = n sin i has been set according to the procedure of Abbe. For the tangential projection corresponding to the axial point, we have

$$\Delta \frac{n \cos^2 i}{p} = \frac{N D''}{\rho_I} = D_I \cos i \cos i' \qquad \Delta \frac{n \cos i \chi_I}{p} = 0$$

and as control

$$\Delta \frac{n \chi_I^2}{p} = -\chi_I \chi_I' D_I.$$

If the axial point represents the diaphragm center and if we desire -- with a narrow diaphragm -- to employ only the laws of the first order to the projection of the extra-axial points, we then need not pay any attention in the individual surfaces to the latter but can calculate them in a manner to be indicated below by application of the complete system. However, if the laws of the second order are to be taken into account, then the values $\tau\zeta$ and the respective magnification coefficients must be known everywhere. We have, for $\tau' K_I'$, similar formulas as for $p' \chi_I'$, and further

$$\Delta \frac{n}{\varsigma} = D_{II} \qquad \Delta \frac{n K_{II}}{\varsigma} = 0$$

as well as for control

$$\Delta \frac{n K_I \chi_I}{\tau - p} = 0 \qquad \Delta \frac{K_{II}}{(\varsigma - q) \sin u} = 0.$$

When u = 0, the value of q' is derived from the equation

$$q' = \frac{N \sin \varphi}{\sin u'},$$

but otherwise the formulas can be utilized unchanged. With $\varphi = 0$ and $y \neq 0$, M and N have infinitely large values where M - N = x. N sin φ has to be replaced by y so that we obtain $D_{II} = 0$ and

$$D_I \cos i \cos i' = \frac{\Delta n \cos i}{\rho_I} = -\frac{n n' \sin \Delta i}{I \rho_I},$$

whereas the formulas remain valid unchanged otherwise.

If the entire system has been recalculated in this manner, we obtain the tangential power of refraction $\mathfrak{D}_I$ of the entire system parallel to the given ray through the formula

$$\chi_I \chi'_I \mathfrak{D}_I = \Sigma \chi_I \chi'_I D_I ,$$

where the magnification coefficients on the left side belong to the first and/or last medium. In the same manner, the value of the sagittal power of refraction $\mathfrak{D}_{II}$ is produced and, if we have also determined the magnification coefficient corresponding to an extra-axial object point, the formula is valid also with the application of the latter which enables us to check the entire calculation. The deduction has been given elsewhere.[1)]

If we now wish to project any desired object point located on the known ray, then the formulas

$$\Delta \frac{n \chi_I^2}{\tau - p} = \chi_I \chi'_I \mathfrak{D}_I \qquad \Delta \frac{n \chi_I K_I}{\tau - p} = 0$$

and the similar ones for sagittal projection are valid.

In the formulas deduced by me for the application of the <u>laws of imagery of the second order</u>[2)], the designations and signs have been selected everywhere so that they correspond to the definitions given above. The formulas can therefore be applied directly with the magnitudes obtained in recalculation and it remains only to briefly discuss the <u>asymmetrical values</u>

$$U = \frac{d}{d\sigma} \frac{1}{\rho_I} \qquad W = \frac{d}{d\sigma} \frac{1}{\rho_{II}}$$

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1. See, e.g., Handbuch der Physiologischen Optik by H. v. HELMHOLTZ, 3d ed., Hamburg and Leipzig 1909.
2. Die reele optische Abbildung. Cf. "Handlingar", Vol. 41, No. 3, 1906.

occurring in them. Here $d\sigma$ is the element of arc of the meridian curve of the non-spherical surface and, in accordance with the given definitions, there are valid without restriction the relations

$$d\sigma = \rho_I d\varphi = \frac{dx}{\sin\varphi} = \frac{dy}{\cos\varphi}$$

With surfaces of revolution, W is determined through the differential quotients of the first and second order and we thus have $\rho_{II} = N$. If a parallel curve of the meridian curve of the non-spherical surface is drawn through the intersection of the normal with the axis, then the element of arc of this parallel curve is equal to

$$(\rho_I - N)\, d\varphi = \sin\varphi\, dM = tg\,\varphi\, dN,$$

which results in

$$W = \frac{(N - \rho_I)\cot\varphi}{\rho_I N^2}$$

The value of U is to be determined separately below for the surfaces of the second degree and for the duplex surfaces. Here we shall merely draw attention to the fact that the magnitudes occurring in the asymmetrical values are algebraic, and not absolute, magnitudes so that the asymmetrical values with equal absolute ordinate magnitude change sign both with a change of sign of the ordinate as well as with inversion of the curve with the ordinate as axis. The latter is true also of the abberation values introduced by me into the Seidel formulas

$$\Phi = \frac{d^2}{d\sigma^2}\frac{1}{\rho_I} \qquad \Omega = \frac{d^2}{d\sigma^2}\frac{1}{\rho_{II}},$$

for which $\Phi = 3\Omega$ in systems of revolution. The value of Φ is obtained from the differential quotient of the fourth order of the meridian curve of the non-spherical surface where $\rho^3\,\Phi = \frac{r^{IV}}{r_o}$.

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Concerning the calculations so far not discussed for <u>surfaces of the second degree</u>, the values of the intrinsic coordinates MN φ have been

determined already above (p. 47) from the equation

$$y^2 = 2\rho x + qx^2;$$

$$\operatorname{tg}\varphi = \frac{y}{\rho + qx}$$

$$M = \rho + e^2 x \qquad N^2 = \rho^2 + e^2 y^2$$

$$M = \frac{e^2 N\cos\varphi - \rho}{q} \qquad N = \frac{\rho}{\sqrt{1 - e^2\sin^2\varphi}}.$$

Since, at $e^2 < -1$, only that part of the curve for which M does not pass through zero is taken into consideration and, at $e^2 > 1$, only one branch of the hyperbola is appropriate; the relations noted there and immediately above agree completely and N receives the same sign as ρ when extracting the root. The relation valid for conical sections

$$\frac{1}{\rho_1} = \frac{\rho^2}{N^3}$$

directly produces

$$U = \frac{d}{d\sigma}\frac{1}{\rho_1} = \frac{3\rho^2}{N^3}\frac{d}{d\sigma}\frac{1}{N} = \frac{3\rho^2 W}{N^2}.$$

It has already been shown above that $\rho^3\Phi = -3e^2$.

In those cases where the axial point of intersection of the oblique rays is located in one of the two media separated by the surface of the second degree, we can select the point of incidence on the surface and then need apply only the formulas indicated. If this condition is not satisfied, the intersection of the conical section with a given ray passing in the plane of the latter is determined in the following manner. If the incident ray, as usual, is determined by the magnitudes su, we then have the quadratic equation

$$2\rho x + qx^2 = (s - x)^2 \operatorname{tg}^2 u,$$

whose roots are

$$x = \frac{A \pm C}{B}$$

and where $A = \rho + s\,\mathrm{tg}^2 u \qquad B = \mathrm{tg}^2 u - q \qquad C^2 = A^2 - Bs^2\,\mathrm{tg}^2 u$

At $B < 0$, the roots have different signs and the conical section therefore represents a hyperbola, the two branches of which are intersected by the incident ray. Since here $|C| > |A|$ and since x must have the same sign as ρ, the lower sign must be utilized for C at positive ρ and vice versa. If $B = 0$, then the roots at $C^2 > 0$ are real with the same sign as A. If here the product ρA is negative, we then have a hyperbola in which the other branch is intersected by the ray at two points. In the contrary case, we must give C the sign opposed to ρ, in order to obtain the numerically smaller root. Imaginary roots correspond to the case $A = 0$ at $B > 0$. At $B = 0$, the one intersection of the ray with the conical section moves into infinity so that a hyperbola is given and the ray is parallel to the asymptote. For the other intersection, there results

$$x = \frac{s^2\,\mathrm{tg}^2 u}{2A}$$

and the intersection belongs to the other branch unless $\rho A > 0$. Finally, the case $C = 0$ is excluded through the conditions $|i| < \frac{\pi}{2}$. This produces the following rules which make it possible to turn over calculation to a routine mathematician which is an important factor in practical calculations. In order for the ray to intersect the surface, we must have $C^2 > 0$ and either $B < 0$ or else $\rho A > 0$ at $B \geqq 0$. The sign of C is to be opposed to that of ρ. However, this last rule has an exception in such cases as may occur, e.g., in dark-field illumination where the numerically larger root must be selected among two roots with the same sign.

In order to finally obtain the <u>intersection of a surface of revolution of the second degree with a ray not intersecting the axis</u>, the equation of the surface is formed in the three-axis coordinate system in which

the Z-axis perpendicularly intersects the already employed X Y-plane by substituting $y^2 + z^2$ for y^2 in the equation of the conical section. Regardless of which characteristics are utilized for the ray, it is always easy to form two equations $y = x_1$ $z = X_2$ where the right side contains only x. If these equations are squared and summated, we then obtain a quadratic equation in x which is to be treated according to a similar scheme as above.

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In calculation with duplex surfaces, the magnitudes x y M N φ ρ ρ_1 are to be employed always only for the meridian curve of the surface itself whereas the machine curve is designated through the respective equation $R = f(\beta)$ as above. Since a passage of R' through infinity is excluded, one point of the meridian curve of the surface is definitely determined through the value of β. This angle is to be calculated as positive in the same direction as φ and must also pass through zero simultaneously with this last angle. In grinding of concave surfaces in which the radii of curvature increase toward the periphery, there may be required, at a large opening, a value of $|\beta| > \frac{\pi}{2}$ also at a moderate value of φ and can be realized mechanically also under certain conditions. Signs and magnitudes of R_o are indicated in grinding for a parallel curve and/or base curve through the relation $R_o = \rho (1 + o)$ and/or $R_o = \rho$. In order to obtain an even approximate idea of the form of the duplex surface, it is preferable to calculate a number of point coordinates in which we start from freely selected values of β and base ourselves preferably on equal intervals in order to facilitate interpolation which may possibly occur later. For the determination of the point coordinates, it is only necessary to know the radius vector and the first derivation. The equations

$$f(\alpha) = c\varphi(\beta) \qquad f'(\alpha)\alpha' = c\varphi'(\beta)$$

produce the values of α and α' corresponding to the respective value of β which are inserted in the equations

$$R - R_0 = R_0 C\varphi(\alpha) \qquad R' = R_0 C\varphi'(\alpha)\alpha'$$

In those cases where the axial point decisive for recalculation is located in one of the two media separated by the duplex surface, the ray can be determined by the selection of the value of β. In order to obtain the tangential radius of curvature and the direct asymmetry of curvature, we then need the derivations of the second and/or third order and must employ for this

$$f'(\alpha)\alpha'' + f''(\alpha)\alpha'^2 = c\varphi''(\beta) \qquad R'' = R_0 C[\varphi'(\alpha)\alpha'' + \varphi''(\alpha)\alpha'^2]$$

$$f'(\alpha)\alpha''' + 3f''(\alpha)\alpha'\alpha'' + f'''(\alpha)\alpha'^3 = c\varphi'''(\beta)$$

$$R''' = R_0 C[\varphi'(\alpha)\alpha''' + 3\varphi''(\alpha)\alpha'\alpha'' + \varphi'''(\alpha)\alpha'^3]$$

It would obviously lead us too far to indicate here the formulas for the various machine types but it may be pertinent also to demonstrate by an example that the calculations are not too complicated. For this, we shall select the most important case in practical application of the general sinus mechanism with standard eccentrics both in the A- and in the B-mechanism by taking into account the general principle that the formulas should make as little demand as possible on the routine calculation. Consequently, we shall also indicate here and there control formulas for early disclosure of any eventual errors of calculation. The machine constants C c ω and the magnitudes $R_0\beta$ are therefore known. We first determine α by means of the equation

$$\sin(\alpha + \omega) = \sin\omega + 2c\cos\omega\sin^2\frac{\beta}{2}$$

after which the equation

$$\sin\alpha = c\sin\beta\,\mathrm{tg}\frac{\beta}{2} + 2\,\mathrm{tg}\,\omega\sin^2\frac{\alpha}{2}$$

is utilized for control and, at small α, for determination of an accurate value in the manner indicated above. Furthermore, if we set $K = R - R_0$,

$$K = 2 R_0 C \sin^2 \frac{\alpha}{2} \qquad R' = \frac{R_0 C c \sin \alpha \sin \beta \cos \omega}{\cos (\alpha + \omega)}$$

and for control

$$U = 1 + \frac{\cos \omega}{\cos (\alpha + \omega)} \qquad R' = R_0 C U \sin \alpha \operatorname{tg} \frac{\alpha}{2} \cot \frac{\beta}{2}.$$

Subsequently,

$$V = \frac{U (U - 1)}{2 \cos^2 \frac{\alpha}{2}} \qquad R'' = R' \cot \beta + R' V \cot \frac{\beta}{2}$$

and as control

$$\alpha' = c \sin \beta (U - 1) \qquad \alpha'' = \alpha' \cot \beta + \alpha'^2 \operatorname{tg} (\alpha + \omega)$$

$$R'' = R_0 C \alpha'' \sin \alpha + R_0 C \alpha'^2 \cos \alpha.$$

The magnitudes U V occurring here are the same which were examined in detail during the determination of the machine constants. A confusion of the former with the similarly designated asymmetrical value need not be anticipated because the respective magnitudes do not occur in one and the same stage of the calculation. The elimination of α''' produces, for any desired mechanisms,

$$\frac{R'''}{R'} = \frac{\varphi''' (\beta)}{\varphi' (\beta)} + 3 \alpha'' \left(\frac{\varphi'' (\alpha)}{\varphi' (\alpha)} - \frac{f'' (\alpha)}{f' (\alpha)} \right) + \alpha'^2 \left(\frac{\varphi''' (\alpha)}{\varphi' (\alpha)} - \frac{f''' (\alpha)}{f' (\alpha)} \right),$$

from which we can derive for the present case, in which the quotient of the derivation of the third order into that of the first order is equal to -1 for all functions, the two formulas

$$R''' + R' = \frac{3 \alpha'' R' (U - 1)}{\sin \alpha} = 3 V \cot \frac{\beta}{2} (R'' - \alpha' R' \cot \alpha)$$

without difficulty.

Further calculation will differ, depending on whether the machine curve represents a parallel curve or the base curve. In the first case, we have

$$\operatorname{tg}(\beta-\varphi)=\frac{R'}{R} \qquad M=\rho+\frac{R\sin(\beta-\varphi)}{\sin\varphi}$$

$$N=\frac{R\sin\beta}{\sin\varphi}-\rho o \qquad y=N\sin\varphi$$

$$x=M-N\cos\varphi=2\rho\sin^2\frac{\beta}{2}+2\rho o\sin\frac{\beta-\varphi}{2}\sin\frac{\beta+\varphi}{2}-K\cos\beta,$$

and, among these formulas, the last offers not only an excellent control but produces the more accurate value at small x. By employing the designations

$$P=1+\sin^2(\beta-\varphi)-\frac{R''\cos^2(\beta-\varphi)}{R}$$

$$G=R^2+2R'^2-RR''$$

we further obtain

$$\rho_1+\rho o=\frac{R}{P\cos(\beta-\varphi)}=\frac{R^3}{G\cos^3(\beta-\varphi)}.$$

For the curvature of the machine curve, there is valid in general

$$\frac{1}{\rho_1+\rho o}=\frac{G}{L^3},$$

in which

$$L^2=R^2+R'^2 \qquad \frac{dL}{d\beta}=\frac{R'(R+R'')}{L}$$

If the designation

$$Q=\frac{d}{d\beta}\frac{1}{\rho_1+\rho o}$$

is introduced, we then have, since

$$(\rho_1+\rho o)\cos(\beta-\varphi)\,d\varphi=R\,d\beta \qquad Q=-\frac{1}{(\rho_1+\rho o)^2}\frac{d\rho_1}{d\beta} \qquad \rho_1^3 U=-\frac{d\rho_1}{d\varphi}$$

for the asymmetrical value

$$U=\left(\frac{\rho_1+\rho o}{\rho_1}\right)^3\cdot\frac{Q\cos(\beta-\varphi)}{R}.$$

Differentiation initially produces

$$Q=\frac{1}{L^3}\frac{dG}{d\beta}-\frac{3G}{L^4}\frac{dL}{d\beta},$$

and we obtain, by employing this expression, the formula system

$$L = \frac{R}{\cos(\beta - \varphi)}$$

$$Q_1 = \frac{1}{L^4}\frac{dG}{d\beta} = \frac{2RR' + 3R'R'' - RR'''}{L^4} \qquad Q_2 = \frac{G}{L^5}\frac{dL}{d\beta} = \frac{R'(R+R'')}{L^5(\rho_1+\rho o)}$$

$$Q_3 = \frac{R(R''' + R')}{L^4} \qquad Q_4 = \frac{R'(R+R'')(R'^2 - RR'')}{L^6}$$

$$U = \left(\frac{\rho_1 + \rho o}{\rho_1}\right)^3 (Q_1 - 3Q_2) = -\left(\frac{\rho_1 + \rho o}{\rho_1}\right)^3 (Q_3 + 3Q_4).$$

If the machine curve represents the base curve, the formulas are basically simple:

$$\varphi = \beta \qquad M = \rho + \frac{R'}{\sin\beta}$$

$$N = R + R'\cot\beta \qquad y = N\sin\beta$$

$$x = M - N\cos\beta = 2\rho\sin^2\frac{\beta}{2} + R'\sin\beta - K\cos\beta$$

$$\rho_1 = R'' + R \qquad U = -\frac{R''' + R'}{\rho_1^3}.$$

In those cases also where the respective axial point is not located in one of the media separated by the non-spherical surface, we can, if the system contains only one non-spherical surface, determine a surface point by the selection of the angle β in order then to search for the ray which passes in the respective field through the axial point and on which the surface point determined by β is located. Whether this method is given preference or whether we search directly for the intersection of the surface with a given ray, will depend in most cases on how many surfaces exist between the axial point and the non-spherical surface. The latter method must be employed in all cases where the optical instrument contains more than one non-spherical surface.

In order to find a ray which passes through a given point of the

non-spherical surface and through a given axial point in another medium, we attempt to determine, if no other ray passing through the respective axial point is already known, initially a rough approximation value by employing the equations valid on the axis where we project the two points in the media where the other point is located. For the sake of simplicity, it shall be assumed here that the light moves in the direction from the axial point toward the surface point and the respective media are considered in calculation as the first and last medium of an optical system where the respective magnitudes are designated by us... u's'.... The axial point is therefore located at the distance s from the first surface, and the distance of the point conjugate on the axis in the last medium of the latter shall be s'_o and the magnification coefficient $\frac{\chi'}{\chi}$. If the axial point represents a real diaphragm center, we have consequently $s < 0$. The vertex of the first and/or last surface represents the starting point of the coordinate system in the respective medium. The coordinates of the surface points shall be x'y'. Through the laws valid on the axis, we determine the point x 0 conjugate to the axial point x'0 and the magnification coefficient corresponding to these points through which the ordinate y is determined. The equation

$$\operatorname{tg} u_0 = \frac{y}{s - x}$$

produces a first approximate value u_o which determines a ray passing through the axial point. The latter is pursued through the optical system by eventually calculating also the magnitudes q' - p' and $\frac{\chi'}{\chi'}$. In this last medium, the ray does not pass through the surface point in the general case. However, we can obviously vary the value of u_o until this is the case, and it is merely a question of saving of time and labor whether we want or do not want to employ the Newton method in the manner to be described

below. If x'y, are the coordinates of the intersection of the refracted ray with the ordinate of the given surface point, we then have

$$y_1 = (s' - x') \operatorname{tg} u'$$

and obtain by differentiation

$$dy_1 = \operatorname{tg} u'\, ds' + \frac{s' - x'}{\cos^2 u'}\, du'.$$

If we further plot a circle through the intersection of the ray with the axis and the center of the circle coincides with the tangential focal point corresponding to the given axial point, it will then become immediately evident that

$$\sin u'\, ds' = (p' - q')\, du'$$

and we obtain, by application of the fundamental equation:

$$\frac{du}{dy_1} = \frac{n' \chi_1' \cos^2 u'}{n \chi_1 \{s' - x' + (p' - q') \cos u'\}},$$

where subsequently, if the angles u are measured in degrees and if E represents the length of the radius of the circle measured in degrees, the equation

$$u_1 = u_0 + E (y_1 - y') \frac{du}{dy'}$$

produces a better approximation value which can be made as accurate as desired by repetition of the procedure.

In the practical application of this method, it should be considered that the calculation of the tangential focal points and of the corresponding magnification coefficients sometimes requires a greater effort than the repetition of the calculation with another ray. We can therefore set, if the given surface point is not located in the proximity of the caustic curve corresponding to the given axial point, p' - q' = 0 and employ for χ_1' the value χ' valid on the axis which makes calculation extremely simple. At a large value of u_0, it is preferable, however, to employ the approximation value

$$p' - q' = 2 (s' - s'_0)$$

in the first calculation and this value results, for u' = 0, through differentiation two times of the above equation for $\frac{ds'}{du'}$.

If we are concerned with directly finding the intersection of the duplex surface with a given ray, then this ray shall be determined by the values s u where the vertex of the surface represents the starting point of the coordinate system. Such a problem presupposes that a number of surface points are already known through previous calculation of the coordinates x y in accordance with certain values of β. The two points through which the ray passes are determined by means of a drawing or by employing the equation of the ray

$$y = (s - x) \operatorname{tg} u$$

by comparing the resulting ordinate values with the given ordinates after insertion of the given abscissa values. In the first calculation, it is best to employ quadratic interpolation and to select first the three points most closely located to the ray for this purpose. If these points are characterized by the magnitudes $\beta_n x_n y_n (n = 1, 2, 3)$, we form in the customary manner the interpolation equation

$$x = x_1 + (\beta - \beta_1)\frac{x_3 - x_1}{\beta_3 - \beta_1} + c(\beta - \beta_1)(\beta - \beta_3),$$

in which c is obtained by substituting $x_2 \beta_2$ for $x\beta$. After the corresponding equation for y has been determined in the same manner, we obtain, by substitution of the values of x and y in the above equation of the ray, a quadratic equation in β which, when solved, produces the first approximation value. If the initially known surface points were not separated too far, then this value will be so accurate that it will be no longer necessary to employ the quadratic interpolation. In the continuation, we can now either carry out linear interpolation in this same manner or else employ the Newton method. In the latter case, it becomes necessary to

calculate, in addition to the magnitudes MN φ necessary for the determination of the coordinates xy, also the radius of curvature ρ_1. The respective differential quotients are obtained, if the machine curve represents a parallel curve of the meridian curve of the non-spherical surface, from the easily derived relations

$$R d\beta = (\rho_1 + \rho o) \cos(\beta - \varphi) d\varphi$$

$$dx = \rho_1 \sin\varphi d\varphi \qquad dy = \rho_1 \cos\varphi d\varphi,$$

whereas dβ = dφ if the machine curve represents the base curve. If we designate the magnitudes determining the first approximation value by β_o $x_o y_o$, we then obtain the values of x and/or y occurring in the differences $x - x_o$ and/or $y - y_o$ by elimination from the equations of the ray and of the tangent. This is equivalent to saying that the value

$$x = x_0 + \frac{\beta - \beta_0}{E} \cdot \frac{dx}{d\beta}$$

and the value of y formed in the same manner are substituted in the equation of the ray. The closer the approximation value utilized, the more closely will this operation coincide with the linear interpolation.

The <u>intersection of a duplex surface with a ray not intersecting the axis</u> is determined in a similar manner. If the equations of the ray have been brought into the form $y = X_1$ $z = X_2$ where the right side contains only x, this produces, through quadrature and summation of these equations, the equation of a hyperboloid of revolution of one sheet, whose secant with the non-spherical surface represents a circle which must contain the desired intersection. It follows from this that the X-coordinate of this intersection is the same as the abscissa of the intersection of the meridian curve contained in the XY plane with the hyperbola

$$y^2 = X_1^2 + X_2^2,$$

which therefore in this calculation takes the place of the ray in the former. The next consequence of this is that the method of quadratic interpolation

leads to an equation of the fourth degree in β. Whether we want to solve the latter or will prefer repeated linear interpolation, is best decided (when the machine curve represents a parallel curve) on the basis of the skill of the routine mathematician. However, if the machine curve represents the base curve, it is preferable in all cases to repeat linear interpolation and/or the Newton method as in the above procedure. The actual difference then consists in the fact that a quadratic equation is to be solved for each new approximation value. However, this can be avoided by neglecting the member of the second order in $\beta - \beta_o$ but it is necessary to examine in each case whether this method is more rapid.

V. Examples of the Application of Duplex Surfaces

It is now merely necessary to show by a few examples that mechanically applicable values for the machine constants can be obtained for cases occurring in practice. For the sake of easier comprehension, we shall consider here only the simplest forms of the duplex curve which will also afford us an evaluation of the capacity of the simplest machine. Non-spherical surfaces have so far been used mainly in two different fields, i.e., first for the purpose of better fusion of rays in one axial point as in non-planate ophthalmoscope lenses and second for improvement of the projection of extra-axial points as in the non-spherical cataract glasses. To the extent in which technology is becoming more familiar with the utilization of such surfaces, it is probable that both purposes can eventually be reached in one and the same optical instrument, perhaps by the application of two non-spherical surfaces where the effect can be distributed both to the axial as well as to the extra-axial projection on both surfaces. For the time being, it would seem more appropriate to treat each of the two purposes separately and we shall therefore discuss examples for the two main types which are characterized by these different purposes separately.

Duplex Surfaces Eliminating Aberration.

Where improvement of the fusion of rays in one axial point is concerned, there is no point, as has already been explained above, to try for a homocentric bundle of rays in the mathematical sense in practical execution and it is preferable to utilize a surface through which such a fusion of rays is obtained that the latter is practically not differentiated from a homocentric bundle in regard to the purpose of the respective instrument. A non-spherical surface which satisfies this condition shall be here designated generally as aberration-eliminating. Since the concept of

non-planateness also implies the satisfaction of the sinus condition and since in general, merely through the change from a spherical to a non-spherical surface, the sinus condition in the elimination of aberration cannot be satisfied, there then do not exist, if we cling to this concept of non-planateness, any non-planatizing surfaces. The expression "aberration-eliminating" selected for this reason comprises as a special case freedom from aberration, i.e., the incident bundle of rays is free of aberration. Aberration-free surfaces are therefore the surfaces of revolution whose meridian curves represent Cartesian ovals and we know that these convert for certain cases into a curve of the second order. In agreement with the discussion above, such duplex surfaces shall be designated as aberration-free duplex surfaces which can be utilized with an accuracy sufficient in practice instead of the exact aberration-free surfaces. There is no doubt that the hyperboloid, once it becomes more easily accessible, will play the most important role among the aberration-free surfaces because the utilization of two plane-hyperbolic lenses with water cooling between them as condenser would constitute an extraordinary increase of the efficiency of the projection apparatus. For this reason and since it is not excluded that the respective aberration-free duplex surface, e.g., in institutes where a duplex machine exists, may be preferred to the hyperboloid for mechanical or economical reasons, this surface is selected here as the first example. This has the further advantage that, because of the simple equation of the hyperbola, a detailed comparison of the two surfaces will require much less effort.

We shall first search for a duplex surface which can replace the so-called non-planate hyperboloid.

By a correction of aberration, there is usually understood, in the

literature of geometrical optics, that state where a peripheral ray passes through the axial image point. The aberration on the axis is here generally not corrected in that the respective aberration value is different from zero and the intermediate rays also intersect the axis in other points. We then speak of zones of aberration. As will be easily understandable, the evolute of the meridian curve of the Fresnel zone of the refracted bundle of rays has in these cases a cusp corresponding to a ray passing between the axis and the given ray and the point in which the given ray touches the evolute is located, in relation to the cusp, on the opposite side both of the axis as well as of the focal plane. This is evident, if the designations $\rho MN\varphi\rho$ are applied to the meridian curve of the Fresnel zone, from the differential quotient

$$\sin\varphi\, dM = (\rho_1 - N)d\varphi$$

already deduced above. In order for a ray to pass through the focal point, there must be, as on the axis, $M = \rho$ and this is possible only when a point $\frac{dM}{d\varphi} = 0$, i.e., $\rho_1 = N$ is located in between which corresponds to an intersection of the evolute with the axis. The latter is possible in turn only when the evolute has a cusp between it and the axial focal point and, after the evolute has intersected the axis, the latter must also intersect the focal plane in order to be touched by a ray passing through the axial cusp if $|\varphi| < \frac{\pi}{2}$ for the latter. We know sufficiently from experience that such a correction is in practice entirely adequate if the opening of the optical system is not too large. For larger openings and when high demands are made on this system, the usual alternate consists in making two different rays intersect the axial focal point. We find in the same manner that in this case the evolute has a second cusp located on the same side of the axis and of the focal plane as the point of contact with the first ray and

then in turn intersects the axis and the focal plane so that the point of contact with the second ray is located on the same side of the axis and of the focal plane as the first cusp. It was stressed above that the cross-section of the caustic surface has an essential and that of the bundle of rays a secondary influence on the quality of the image. It follows from this that the existence of the cusps on the evolute of the meridian curve of the Fresnel zone of the bundle of rays represents precisely the nature of the correction of the aberration and that one cusp corresponds to the usual but two to greater demands. To this needs to be added only that it is not indifferent on what part of the evolute the cusp is located. If only one cusp exists, then the effect of the latter is obviously much less when it is located in the immediate proximity of the axial focal point or in the most peripheral part of the evolute as when it has a medium position and the same is true -- all things being equal -- also for the existence of two cusps.

There are infinitely many surfaces which refract, at prescribed vertex and radius of curvature of the latter, a given ray issuing from a given object point so that it passes through the axial image point in the second medium. The optical length from the object point to the axial image point changes with the position of the intersection of the surface with the incident ray. Only for the aberration-free surface and for those surfaces which have a contact of the first order with the former at the intersection with the incident ray, is the optical length on the refracted ray equal to that on the axis. That this behavior must represent an advantage may be expected a priori and is determined in the following manner.

If we construct an evolvent intersecting a symmetrical evolute which has no other cusps than the axial, we then find that the evolvent is divided

in three parts through two cusps touching the evolute. From the intersection with the axis, the evolvent proceeds initially with the concave side turned toward the cusp until it encounters the corresponding branch of the evolute where it turns back with the formation of a cusp and then intersects, in proceeding further, the other branch of the evolute with the convex side turned toward the cusp. The evolute which touches the evolvent in the cusp of the latter therefore has a cusp between the points of contact and it will be easily seen that this constitutes a generally valid rule for the evolute of the meridian curve of the Fresnel zone under the conditions determined for optical instruments. The same is true also of the circumstance that two branches joining in a cusp -- branches either of the evolute or of the evolvent -- turn the convex side toward each other. If now, on a ray with finite inclination and passing through the axial focal point, the optical length from the object point is the same as on the axis, then the meridian curve of the evolvent of the evolute of the Fresnel zone passing through the axial cusp of the evolute must intersect itself in the same cusp and have the respective ray as its normal under these circumstances. However, this is only possible if the evolvent has two cusps on both sides which is equivalent to three points of contact with the evolute and therefore requires two pairs of symmetrical evolute cusps in addition to the axial cusp. Since further the point of contact of the given ray with the evolute must be located on the same side of the axis and of the plane perpendicular to the axis, and passing through the axial image point, as the evolute cusp corresponding to the lesser ray inclination, there consequently must also exist a ray with lesser inclination which passes through the axial focal point. It follows from this that an eccentric osculation of the first

order with an aberration-free surface on each side of the axis produces two evolute cusps and two rays passing through the axial focal point.

In an osculation of the second order, the evolute must also return to the axial focal point and here touch the given ray which produces at this point in the corresponding evolvent a third cusp which conditions in turn a third double-sided evolute cusp. In general, with increasing ray inclination, the lateral aberration of a ray changes sign when the latter passes through the axial focal point. However, since this is not the case when the evolute passes simultaneously through the focal point, we can therefore, in an osculation of the second order, regard the given ray geometrically as two coinciding rays passing through the axial focal point. If a centric osculation of the fourth order is added to this, then the centric cusp of the evolute is touched in five points by the axis, or it is possible to create, through variation of the respective constants, on the evolute one more double-sided cusp in the proximity of the axial cusp.

Based on these findings, it was evident that a relatively large opening could be made the basis of the calculations without any risk of failure. I therefore selected a priori that point of the hyperbola as point of osculation for which the ordinate has the same value as the radius of curvature of the vertex. If this value is assumed as equal to 1 and if a refraction index of 1.53 is selected, then

$$x = 0{,}30525 \qquad y = 1 \qquad \varphi = 33°{,}169$$

$$M = 1{,}9253 \qquad N = 1{,}8278 \qquad \rho_1 = 6{,}1064$$

for this point and

$$\rho = 1 \qquad \frac{\rho^3\Phi}{3} = -e^2 = -2{,}3409.$$

for the vertex.

The first experiment with an osculation of the first order already led to a relatively satisfactory result. This first calculation was concerned with the simplest duplex machine with crossed cylinders and with an evolvent eccentric of the first order in the A-mechanism at $\omega = 0$ so that the equation of the machine curve could be written in the form

$$R - R_0 = c_0(1 - \cos\alpha) + c_1(\alpha - \sin\alpha) \qquad \sin\alpha = c(1 - \cos\beta)$$

The values $c = 1$, $o = 0.25$ were selected so that the radius of curvature of the abrading cylinder then constituted one-fourth of the radius of vertex of the surface. This produced

$$c_0 = 1{,}045\,32 \qquad c_1 = -2{,}252\,24.$$

A plane-convex lens provided with this duplex surface and turned with the plane surface to the light showed, for light with a parallel incidence, the following lateral aberration of the various rays:

β	η
$-10°$	$-0{,}000\,957$
$-20°$	$-0{,}005\,428$
$-30°$	$-0{,}007\,502$
$-40°$	$+0{,}005\,197$
$-50°$	$+0{,}016\,07$
$-60°$	$-0{,}076\,03$

Here η is the ordinate of the intersection of the refracted ray with the focal plane and the sign of the ordinate is referred to a positive value of the ordinate of the intersection with the surface as will be seen from the sign of the angles β. Therefore

$$\beta = 54°{,}397 \qquad \alpha = 24°{,}698 \qquad \rho_1 = 4{,}1700.$$

for the point of osculation.

In order for a centric osculation of the fourth order to exist, it would be necessary that $c^2 c_o = e^2$. It is therefore shown by the value of c_o that the surface is located in the proximity of the axis between the hyperboloid and the sphere osculating the latter in the vertex. In agreement with this, the aberration of the rays passing in the proximity of

the axis is also positive. The first change of sign of the latter corresponds to the ray intersecting the axial focal point which must be located between the axis and the ray refracted in the point of osculation. The relative high value of the aberration of the most peripheral ray results from the difference of the radii of curvature.

In order to judge the value of such a duplex surface, it is not sufficient, however, to know the difference from the hyperboloid but we must also take into account the difference from the sphere. In a spherical surface, the occurrence of total reflection conditions a maximum of the ordinate of the surface point at a value of 0.6536 and the corresponding value of η is -1.8128. The ordinate of the surface point corresponds in the duplex surface to a value of $\beta < 40^{\circ}$.

The favorable result justified the expectation that the experiment with an eccentric osculation of the first order at $\omega = 0$ and by employing a standard eccentric in the A-mechanism would also be successful. In the equation of the machine curve

$$R - R_0 = c_0(1 - \cos\alpha) \qquad \sin\alpha = c(1 - \cos\beta)$$

there is continually written c_o for the product R_oC because the mechanical applicability of the corresponding A-eccentric results immediately from this value. Certain experiences in the preceding calculations made it probable that a larger radius of curvature of the abrading cylinder would be of advantage so that a value of $o = 2$ was selected. There resulted

$$c_0 = 0{,}989\,16 \qquad c = 1{,}6212 \qquad c^2c_0 = 2{,}5098$$

and, for the point of osculation,

$$\beta = 42^{\circ}{,}583 \qquad \alpha = 25^{\circ}{,}310 \qquad \rho_1 = 7{,}107.$$

Consequently, the hyperboloid lies here both in the proximity of the axis as well as in the proximity of the points of osculation between the duplex surface and the sphere osculating the two surfaces in the vertex.

Under the same supposition as above, the lateral aberration η was calculated for the various refracted rays. These values are listed below together with the ordinates y of the surface points in which the rays are refracted:

β	y	η
10°	−0,178 21	−0,000 61
20°	−0,377 46	−0,002 57
30°	−0,615 57	−0,000 21
40°	−0,910 92	+0,005 42
45°	−1,089 66	−0,013 68

Here also the double change of sign manifests that a ray exists between the axis and the ray refracted in the point of osculation which passes through the axial focal point and that, correspondingly, two pairs of symmetrical cusps are found on the evolute of the meridian curve of the Fresnel zone of the refracted bundle of rays. As will be seen from the values demonstrated, the correction is so satisfactory that the possibility of differentiating this duplex surface from the hyperboloid in practical application is very minor.

Partly in order to show that it is possible to obtain even much better results and partly in order to demonstrate the manifold applicability of the duplex method, we have carried out further calculations of machine constants.

Initially, there was obtained an eccentric osculation of the second order through application of the general sinus mechanism where we therefore had to set $\frac{\sin(\alpha + \omega) - \sin \omega}{\cos \omega}$ instead of $\sin \alpha$ in the last equation above of the machine curve. The first experiment was effected with $o = 0.25$ and, since this was satisfactory, o was then varied for the purpose of achieving a simultaneous eccentric osculation of the fourth order. The main results are listed in the table below and in the sequence in which the calculations

were carried out:

o	ω	$c^2 c_0$	c
0,25	$-31°,769$	2,3029	2,252
1,0	$-14°,323$	2,4045	1,207
0,6	$-24°,857$	2,3819	1,834
0,4	$-29°,048$	2,3503	2,094
0,35	$-29°,964$	2,3359	2,148
0,37	$-29°,663$	2,3447	2,1326

Since $c^2 c_0$ must have the value of 2.3409 in order for us to obtain the osculation of the fourth order in the vertex, there would be no point in carrying the calculations out further. If this value has been reached approximately, then the correction may be as satisfactory as at mathematical equality which can be decided only through the time-consuming comparative examination of the lateral aberration of the various rays. This is specifically the case when the special cusp characterizing the centric osculation of the fourth order on the evolute on the meridian curve of the Fresnel zone of the refracted bundle of rays has become decomposed into three cusps which is made manifest by the sign of the lateral aberration of the rays passing nearest to the axis. Based on the machine constants

$$o = 0,37 \qquad |c_0| = 0,51554 \qquad c = 2,1326 \qquad \omega = -29°,663$$

we therefore first calculate the lateral aberration of the various rays in the manner indicated above. The values are listed in the following table together with the respective ordinates of the surface points:

β	y	η
5°	−0,087 366	0
10°	−0,175 29	0
15°	−0,264 26	0,000 07
20°	−0,354 56	0,000 25
25°	−0,446 45	0,000 41
30°	−0,540 11	0,000 47
35°	−0,635 80	0,000 28
40°	−0,733 95	−0,000 24
45°	−0,835 14	−0,000 46
50°	−0,940 14	−0,000 07
55°	−1,050 34	−0,000 50
60°	−1,167 90	−0,008 85

For the calculations, we employed seven-place logarithms until we obtained a value at a certain place where the accuracy of this value was such that greater accuracy could not be obtained through continuation with a higher number of places than five. The values obtained for the lateral aberration of the two rays with the smallest inclination were positive but so small that the sign was uncertain, in view of the number of places employed, and are therefore designated in the table with O.

As is shown by the table, the lateral aberration is everywhere so small that a greater correction, even though mathematically possible in all probability, would most likely be physically not noticeable. The change of sign of the lateral aberration indicates that, in addition to the ray refracted in the point of osculation, another ray passes through the axial focal point which is possible only when three cusps exist on the evolute of the meridian curve of the Fresnel zone of the refracted bundle of rays between the point of contact with the first ray and the axial focal point. Since moreover $c^2 c_o > e^2$, an also infinitely small negative value of y must correspond to an infinitely small negative value of η or, in other words, the cusp of the axial evolute has become decomposed into three cusps so that the evolute has a total of not less than nine cusps.

In order to demonstrate the degree of similarity of the duplex surface with the hyperboloid, I have calculated the table shown further below. The value of the abscissa belonging to each value of θ was substituted in the equation of the hyperboloid whereupon the corresponding values of $y\varphi NM\rho$, belonging to the hyperboloid were calculated. These values are designated in the table by H whereas D indicates the values belonging to the duplex surface at the same abscissa. The numerical

calculations were carried out by routine mathematicians in my laboratory and a sufficient number of control formulas guaranty the accuracy of the results. However, since I have neither verified myself that there is not an error of the last place in a logarithm nor performed a calculation of the degree of accuracy which would represent an extremely time-consuming effort, it is probably not possible to determine with certainty from the differences of the ordinates that the meridian curves of the two surfaces intersect in accordance with the four changes of sign of the curves in four points which could have been decided only through more detailed calculations. However, we may conclude, because of the great number of calculations, that the differences are not greater than the maxima indicated in the table. Since the distance of the two surfaces from each other (if they touch in the vertex) is approximately equal to the difference of the ordinate multiplied by sin φ, the greatest difference amounts to about 1/10,000 of the radius of curvature of the vertex. In a plane-convex lens with a radius of curvature of the vertex of 10 cm and with a diameter of 21 cm, the largest deviation of the form of the duplex surface from that of the hyperboloid would therefore be indicated by a distance of about 0.01 mm and, if this lens is struck on the plane surface by axial-parallel light, the maximum lateral aberration of a ray amounts to 0.06 mm. Here the angles β and α have, in accordance with the point of osculation, the value of 52.749° and/or 43.349° so that the machine constants are entirely satisfactory from the mechanical viewpoint.

However, duplex surfaces can also be obtained in another manner in these surfaces which are exceptionally appropriate for replacing the hyperboloid. For example, the duplex surface first described above which was calculated for the application of an evolvent eccentric in the A-mechanism

Table for comparison of the non-planate hyperboloid with a duplex surface.

x		y	φ	N	M	ρ_1
0,003 79	D	0,087 37	4°,967 70	1,008 90	1,008 92	1,026 93
	H	0,087 17	4°,956 97	1,008 86	1,008 87	1,026 81
0,015 20	D	0,175 29	9°,748 18	1,035 31	1,035 57	1,109 55
	H	0,175 22	9°,743 80	1,035 55	1,035 57	1,109 72
0,034 14	D	0,264 26	14°,184 33	1,078 41	1,079 67	1,253 59
	H	0,264 26	14°,181 25	1,078 65	1,079 91	1,255 00
0,060 44	D	0,354 56	18°,168 5	1,137 09	1,140 83	1,469 71
	H	0,354 65	18°,162 4	1,137 74	1,141 48	1,472 72
0,093 76	D	0,446 45	21°,647 8	1,210 19	1,218 62	1,773 98
	H	0,446 44	21°,632 3	1,211 03	1,219 48	1,776 04
0,133 80	D	0,540 11	24°,619 5	1,296 48	1,312 53	2,186 94
	H	0,540 21	24°,607 0	1,297 40	1,313 42	2,183 65
0,180 42	D	0,635 80	27°,119 6	1,394 78	1,421 84	2,729 9
	H	0,635 99	27°,116 8	1,395 30	1,422 35	2,716 5
0,233 04	D	0,733 95	29°,211	1,503 92	1,545 72	3,418 1
	H	0,734 08	29°,219	1,503 80	1,545 52	3,400 9
0,291 75	D	0,835 14	30°,968	1,623 01	1,683 40	4,263 8
	H	0,835 24	30°,979	1,622 70	1,682 95	4,272 6
0,356 78	D	0,940 14	32°,455	1,751 92	1,835 00	5,328 4
	H	0,940 35	32°,459	1,752 12	1,835 18	5,378 8
0,428 58	D	1,050 34	33°,704	1,892 82	2,003 25	6,955 9
	H	1,050 46	33°,707	1,892 91	2,003 25	6,782 5
0,508 50	D	1,167 9	34°,666	2,053 3	2,197 4	11,143 3
	H	1,167 8	34°,774	2,047 5	2,190 3	8,584 0

at $\omega = 0$, can be improved through variation of the machine constants o and c until a centric osculation of the fourth and an eccentric osculation of the second order can be obtained with any desired degree of accuracy. Without including a table of all calculations, I shall indicate here only the final result. If we set o = 0.65 and c = 1.8325, there then result

$$c_0 = 0{,}696\,998 \qquad c_1 = -0{,}687\,384 \qquad c^2 c_0 = 2{,}34055 \qquad \rho_1 = 6{,}1058,$$

and the values

$$\beta = 49^{\circ},722 \qquad \alpha = 40^{\circ},375,$$

correspond to the point of osculation so that the machine constants are entirely satisfactory from the mechanical viewpoint also in this case.

Moreover, the duplex surface calculated above for application of a standard A-eccentric for $\omega = 0$, can be improved also through reducing the value of o where both the product $c^2 c_o$ as well as the radius of curvature on the point of osculation decreases, we thus obtain for o = 1.35:

$$c = 0,745\,95 \qquad c_0 = 4,363\,33 \qquad c^2 c_0 = 2,4279 \qquad \rho_1 = 6,244,$$

where the values for the osculation β and/or α are 45.065° and/or 12.655°. Although the height of the eccentric in the A-mechanism is large, it is still mechanically applicable. If we continue further on this path, it is possible to come theoretically even closer to the optical ideal but c_o then rapidly rises to mechanically disadvantageous values. For example, at O = 1.25, there results the high value of c_o = 10.883 which is presumably applicable only at a very small radius of curvature of the vertex. However, with this we obtain

$$c^2 c_0 = 2,3964 \qquad \rho_1 = 6,039.$$

If we are able to freely select the diameter of the abrading cylinder, we can then obtain duplex surfaces in various ways which will replace the hyperboloid. However, this raises the question of whether this is also possible if we are to grind with a certain given surface and especially if this surface represents a plane because the latter offers particular mechanical advantages as was explained above (p. 66).

If the evolvent eccentric is utilized in the A-mechanism, there then result unsuitable values for the base curve as machine curve. The latter can be improved by the application of the general crank mechanism as B-mechanism but, since this produces an unnecessarily complicated device,

we investigated only the case K = 1 which did show better results. However, since the latter did not seem sufficiently satisfactory to me, we turned again to the general sinus mechanism. The figures in the table of machine constants previously given (p. 178) made it seem probable that an attempt with k = 0 would produce no advantage. For k = 1, i.e., by employing the machine curve

$$R - R_0 = c_0(1 - \cos\alpha) \qquad \frac{\sin(\alpha+\omega) - \sin\omega}{\cos\omega} = c\left(\frac{1}{\cos\beta} - 1\right)$$

as base curve, however, there resulted for the eccentric osculation of the second order

$$c_0 = 0{,}29638 \qquad c = 2{,}79396 \qquad \omega = 8°{,}9008 \qquad c^2c_0 = 2{,}3136,$$

whereas

$$\beta = \varphi = 33°{,}169 \qquad \alpha = 34°{,}892$$

in accordance with the point of osculation. The machine constants are therefore excellent from the mechanical viewpoint and the small difference $e^2 - c^2c_0$ signifies that the surface can replace in practice the hyperboloid from the optical viewpoint. It is moreover possible to obtain the simultaneous centric osculation of the fourth order through variation of k. If such a machine is constructed only for the grinding of these duplex surfaces, then a carriage is not necessary in the B-mechanism because the crank mechanism, as explained above (p. 54), can be replaced by a closed linkage consisting of a sphere integrated with the A-axis and resting on the cylinder linked to the B-axis where this cylinder degenerates into a plane at k = 1. When grinding such duplex surfaces with a different radius of curvature of the vertex, we need only change the distance of the abrading plane from the B-axis and the A-eccentric accordingly. It will be difficult to decide a priori whether a duplex surface produced in this or some other manner will be able to displace the directly ground

hyperboloid on the market through mechanical and economical considerations.

If I have thus shown that there exist at least four different duplex surfaces capable of being produced with simple means which can replace the non-planate hyperboloid, I have simultaneously also demonstrated the multifold applicability of the duplex method. The difference of the different machine curves is illustrated most easily by a comparison of the parallel curve of the hyperbola at a small value of o with a base curve which possesses a point of inflection within the section which is utilized.

After this, let us examine as the next example an aberration-eliminating duplex surface with a point of inflection. The exact aberration-eliminating surface along each ray can be constructed geometrically point-by-point in the following manner according to the method sketched by Huygens. To a given optical system of revolution of m - 1 surfaces, a last surface is added for which the radius of curvature of the vertex and the locus of the vertex are preselected and whose form must be such that a given axial point located in the first medium is projected homocentrically through the entire system which thus consists of m surfaces. A necessary and adequate condition for this is that the optical length on each ray has one and the same value. The optical length existing along an arbitrarily selected ray between the surface with the order number ν and that with the order number $\nu + 1$ can be indicated in the form

$$n'_\nu q'_\nu - n_{\nu+1} q_{\nu+1}$$

For the first and/or last medium, the members $- n_1 q_1$ and/or $+ n'_m q'_m$ are added. After summation, we obtain, by employing the invariant designation, the condition of freedom of aberration represented by the optical length in the form

$$\Sigma \Delta nq = \Sigma \Delta n s_0 .$$

if $s_o s'_o$ represent the paraxial lengths of intercept.

Since the locus of the vertex and the radius of curvature of the vertex of the last surface are given, the right side is known. We then begin by laying off on the ray refracted in the next to the last surface, and therefore belonging to the next to the last medium, such a distance that the optical length from the object point to the end point of this section is equal to the optical length on the axis. In Fig. 5, let ED be the ray refracted on the surface m - 1; E = the point belonging to this surface in which the refraction takes place; O = the vertex of the last surface; and A = the image point.

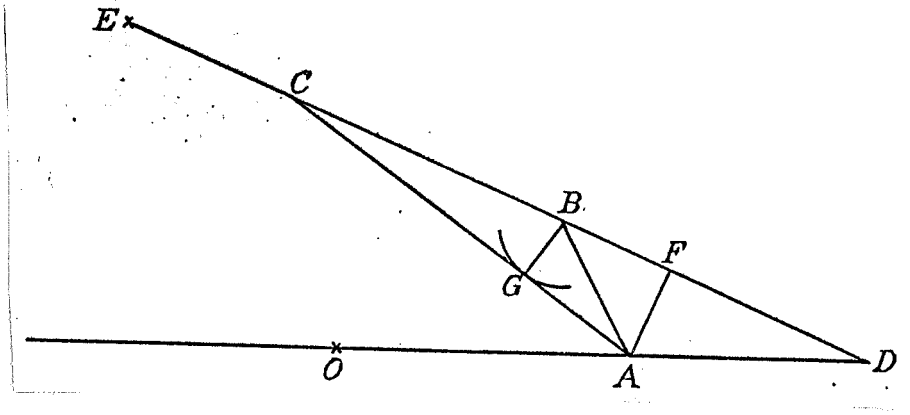

Fig. 5.

If the optical length on the oblique ray from the object point to the point E is designated by L, then

$$L = \sum_1^{m-1} \Delta nq - n'_{m-1} q'_{m-1} = \sum_1^{m-2} \Delta nq - n_{m-1} q_{m-1},$$

and we lay off the section

$$EB = \frac{\Sigma \Delta n s_0 - L}{n_m}$$

on the ray ED. The optical length from the object point to the point B located in the next to the last medium is then the same as to the point A located in the last medium. If we now draw the normal AF on the ray ED from this point and draw a circle with the radius $BG = AF \cdot \frac{n'_m}{n_m}$ with B

as its center, then the intersection C of the tangent of this circle passing through A with the ray ED represents the respective point of the last surface and CA is consequently the ray refracted in the latter. Because of the similar triangles, we have $n_m \cdot CB = n'_m \cdot CA$ and, since the optical length from the object point to the point C is equal to $\sum \Delta ns_o - n_m \cdot CB$, this length to the point A is the same on the refracted ray as on the axis.

In the case where the image point is infinitely distant and all rays must therefore issue axial-parallel from the last surface, the optical length is measured as far as an arbitrary plane intersecting the axis perpendicular in the last medium. However, in this case, the above geometrical construction fails us but the corresponding trigonometric calculation presents no difficulty.

In order to determine trigonometrically the point C as well as the direction of the normal and the tangential radius of curvature in the former, we need to determine first, when concerned with the surface m - 1, the coordinates x,y, of the point B referenced to O as starting point of the coordinate system. If EB is here designated as X and calculated in the manner already indicated, we then have, if xy designate the coordinates of the point E referenced to the vertex of the surface m - 1 as starting point,

$$x_1 = x + X\cos u' - d \qquad y_1 = y - X\sin u',$$

where d represents the distance of the point O from the starting point of the coordinate system. In the calculation of the last surface, let us temporarily designate the angles BCA and BAC with δ and/or ε. We then obtain

$$\sin\varepsilon : \sin(\varepsilon + \vartheta) = n' : n = \sin i : \sin i'.$$

Since moreover $\delta = u' - u = i - i'$, as is evident from the figure, this results in

$$\sin i \sin(\varepsilon + \vartheta) = \sin\varepsilon \sin(i - \vartheta)$$

and from this

$$\operatorname{tg} i = -\operatorname{tg}\varepsilon,$$

and this last equation is unique under the condition $|i| < \frac{\pi}{2}$. Since therefore the angle OAB is in the figure equal to $u' - i = u - i'$, we obtain i' from the equation

$$\operatorname{tg}(u - i') = \frac{y_1}{s' - x_1}$$

and then successively and in the customary manner i φ and u'. Furthermore:

$$q' = \frac{(s - s')\sin u}{\sin(i - i')} = -\frac{y_1 \sin i'}{\sin(i - i')\sin(u - i')},$$

where the first of these expressions is the simpler one and is employed advantageously as soon as s has not too high a value. The last expression provided for this case results from the application of the triangle ABC where AB = $\frac{y_1}{\sin(u - i')}$. Through the usual formulas

$$N = \frac{q' \sin u'}{\sin\varphi} \qquad M = s' - \frac{q' \sin i'}{\sin\varphi}$$

$$x = M - N\cos\varphi \qquad y = N \sin\varphi$$

we know the magnitudes necessary for the calculation of an osculation of the first order and those for the construction of the surface point. In order to obtain also the radius of curvature, the value of ρ is desired either in the usual manner, if s is not too large, from the difference q - p known from the preceding surface or else from the expression

$$p = p'_{m-1} - X + \frac{n' q'}{n},$$

from which ρ_1 is derived from the formula

$$\Delta \frac{n \cos^2 i}{p} = -\frac{n n' \sin \Delta i}{\rho_1 I}$$

in which we must set $\rho' = q'$.

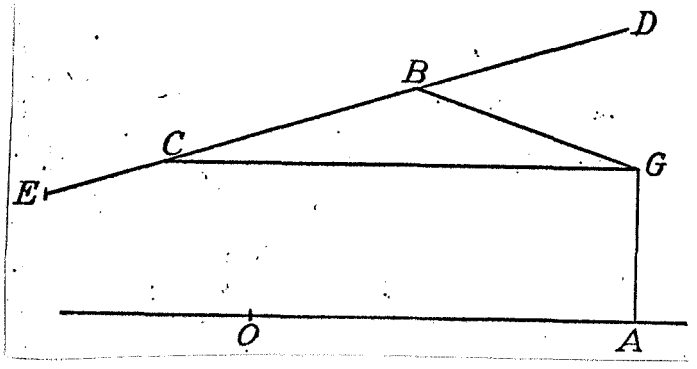

Fig. 6.

The case of an infinitely distant image point is illustrated most simply by Fig. 6 where O, E and C have the same significance as in Fig. 5 and ED therefore represents the ray refracted in the next to the last surface. A is an arbitrarily selected axial point in the last medium and the optical length from the object point to the former is equal to the optical length to the point B located in the next to the last medium. CG is the ray refracted in the last surface which intersects the line AG perpendicular to the axis in the point G. This then gives $n_m \cdot CB = n'_m \cdot CG$ and proves in the same manner as above that the angle BGC = -i. Since u' = 0, we have i - i' equal to -u and consequently

$$n \sin i = n' \sin (i + u),$$

from which results

$$\operatorname{tg} i = \frac{n' \sin u}{n - n' \cos u}.$$

If OA is designated by E and x, y, are the coordinates of the point B as above, we then obtain the coordinates of the surface point C by means of the formulas

$$y = y_1 + (E - x_1) \operatorname{tg} i = (s - x) \operatorname{tg} u,$$

after which all required values are determined through the usual formulas. Where we are concerned only with calculation but not with explanation by means of a figure, it is simplest to make E = 0.

In order to apply the application of this method to calculation of an aberration-eliminating duplex surface, let us select an example which makes high demands on the duplex method. If we wish to satisfy, in reverse projection at the same magnitude by a simple lens, also the sinus condition for the peripheral ray when one surface is spherical, then the other surface calculated in the above manner receives a point of inflection of the meridian curve which will be located within the optically effective part at an adequately large opening. In order to produce such a lens, we must construct, in the general case and after experimentally effected deflection, the respective point of the non-spherical surface and the ray refracted in the former in the manner just described until that value of the deflection has been found which satisfied the sinus condition. However, these calculations may be replaced, if the lens has a sharp periphery and the peripheral ray is selected for the elimination of the sinus condition, in the following manner by solving a cubic equation. That it is best, even when the lens cannot be optically exploited all the way to the sharp periphery, to select the peripheral ray for the satisfaction of the sinus condition will be seen from the following. Similarily acting lenses are in use for other magnification as "non-planate ophthalmoscope lenses" in the methods of non-reflecting ophthalmoscopy developed by me. Since the designation "non-planate" has already been introduced for similar lenses, although it should actually only designate lens in which both the aberration is eliminated and the sinus condition is satisfied for any arbitrary inclination of the ray, it should be employed here also. We are therefore concerned with <u>the construction of a duplex lens with one spherical surface non-planate for the magnification coefficient-1</u>.

In order to find the deflection corresponding to an arbitrary

magnification coefficient by means of the cubic equation, we can proceed in the following manner. For the ray refracted at the sharp periphery of the lens, we select q_1 and u_1 which is equivalent to the choice of scale and lens aperture. Through the sinus condition, q'_2 and u'_2 result from the prescribed magnification coefficient. The thickness of the lens is determined by the condition that the optical length on the peripheral ray shall be the same as on the axis. If the lengths of intercept otherwise designated by s are designated instead with S and if n is the refraction index of the lens, then this condition is written

$$-q_1 + q'_2 = -S_1 + nd + S'_2,$$

whereas we have on the other side

$$-q_1 \cos u_1 + q'_2 \cos u'_2 = -S_1 + d + S'_2$$

We obtain from this for the thickness of lens

$$d = \frac{q'_2(1-\cos u'_2) - q_1(1-\cos u_1)}{n-1}.$$

The two equations

$$\Delta \frac{n_\nu \chi_\nu}{S_\nu} = 0 \qquad (\nu = 1, 2)$$

result in

$$\frac{S'_1}{S'_1 - d} = \frac{S_1 \sin u_1}{S'_2 \sin u'_2}.$$

because of the satisfaction of the sinus condition. On the other hand, we have, when using

for abbreviation

$$a = q'_2 - q_1 - nd$$

$$S'_2 = S_1 + a,$$

which furnishes

$$S'_1 = \frac{S_1 d \sin u_1}{S_1(\sin u_1 - \sin u'_2) - a \sin u'_2}$$

The paraxial projection in the first surface signifies

$$\frac{n}{S'_1} = \frac{n-1}{\rho_1} + \frac{1}{S_1},$$

and if xy represent the coordinates of the intersection of the XY-plane with the periphery of the plane in the standard coordinate system whose starting point is shifted to the intersection of the first spherical surface with the axis, we then have

$$x = S_1 - q_1 \cos u_1 \qquad y = q_1 \sin u_1 \qquad (\rho_1 - x)^2 + y^2 = \rho_1^2$$

and obtain, through elimination of $S'_1 S_1 \rho_1$ and y from the last five equations, the cubic equation

$$x^3 n(1-k) + x^2\{nq_1 \cos u_1 (1-k) - d(2n-1) - nka\} +$$
$$+ xq_1\{nq_1 \sin^2 u_1 (1-k) - 2d \cos u_1 (n-1)\} + q_1^2 \sin^2 u_1 \{nq_1 \cos u_1 (1-k) - d - nka\} = 0.$$

in which k was substituted for $\frac{\sin u'_2}{\sin u_1}$. If k = -1 and if we designate $q'_2 = -q_1$ and/or $u'_2 = -u_1$ by q and/or u, then this equation can be brought into the form

$$x^3 \cdot \frac{n(n-1)}{1-\cos u} - x^2 q(3n-1) + xq^2(n-1)(n + n\cos u + 2\cos u) - q^3 \sin^2 u (n+1) = 0$$

For n = 1.53 and u = 26.6°, there then results $\frac{x}{q}$ = 0.24572 and, if the absolute value of the radius of curvature of the vertex of the second surface is selected as unit,

$$\rho_1 = 1{,}8290 \qquad d = 1{,}3765 \qquad \rho_2 = -1 \qquad S_1 = -2{,}2347.$$

Based on these values, I have calculated the coordinates of the respective points on the exact aberration-eliminating surface for a different inclination of ray and listed them in the following table:

u_1	x	y
− 4°	−0,015 65	0,177 05
− 8°	−0,061 02	0,355 03
−12°	−0,130 90	0,531 58
−15°	−0,205 76	0,665 29
−18°	−0,270 39	0,804 50
−20°	−0,324 46	0,904 00
−22°	−0,382 33	1,015 38
−24°	−0,444 97	1,153 32
−25°	−0,478 72	1,245 72
−26°	−0,514 32	1,381 82
−26°,6	−0,532 22	1,544 12

By using these values, Fig. 7 has been drawn in order to give a picture of the form of the lens and of the demands made on the duplex surface to be calculated. Strokes indicate the principal paraxial points and crosses the center of curvature of the spherical as well as the center of curvature of the vertex of the non-spherical surface. At the periphery of the lens, this produces for the latter surface at negative u_1

$$M = -2269{,}038 \qquad N = -2268{,}506 \qquad \varphi = -0°{,}0390 \qquad \rho_1 = +0{,}83522.$$

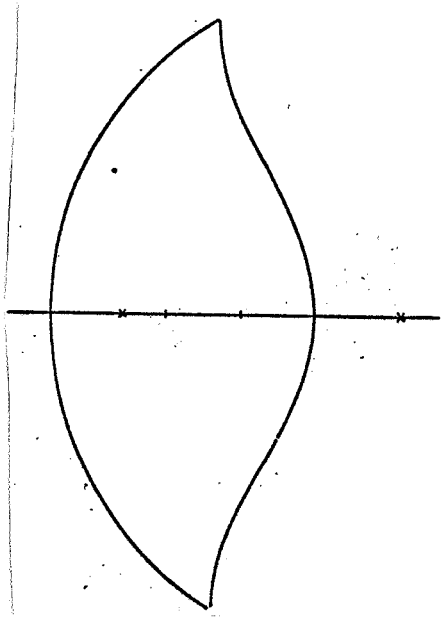

Fig. 7

It is here evident that the surface normal is nearly parallel to the axis and the degree of curvature is negative because the tangential but not the sagittal radius of curvature has changed signs. The corresponding duplex surface can consequently still be ground with a cylinder but, since o may not exceed the absolute value of $\frac{\rho}{\rho'}$, the possibility of satisfying a prescribed condition through variation of this magnitude is correspondingly limited a priori. To this should be added, as shown by the figure, that the radius vector at the periphery of the lens is very appreciably prolonged and intersects the tangential plane of the surface at a relatively small angle so that high values of the differential quotients in the polar equation of the machine curve result. If I utilize in spite of this, the above values valid for a point of the periphery of the lens for the osculation of the duplex surface, then this is not done merely in order to calculate a practically advantageous form of lens (which will be discussed further below) but in order to select an example which makes high demand on the duplex method.

We will therefore have to investigate whether it is possible to obtain with simple technical means an osculation of the first and/or second order on the periphery of the lens and perhaps also a simultaneous centric osculation of the fourth order. In order to satisfy this last postulate, we need $\frac{\Phi}{3} = 2.0651$ at a negative radius of curvature of the vertex as is shown through the respective Seidel formula in the form indicated by me.

The first attempt proved that a rather satisfactory result is obtained already with the simple sinus mechanism, i.e., at $\omega = 0$. With a positive radius of curvature of the vertex, there resulted for $O = 0.6$, at eccentric osculation of the first order,

$$\rho_1 = -0{,}83893 \qquad c^2 c_0 = 1{,}8548,$$

and the eccentric osculation of the second order can be achieved through a minor reduction of o at any desired degree of accuracy as proved by calculations which are not given here. However, the value of $c^2 c_o$ does deviate from the desired value but must be considered as relatively favorable since only three machine constants are available for the satisfaction of four conditions. The value of α corresponding to the point of osculation is 75.957^o which has to be regarded as a disadvantage from the technical viewpoint. However, if we consider that, as will be seen from Fig. 7, the speed of the A-carriage at very large β must be very high in relation to the speed of rotation around the B-axis, we can predict that it will be scarcely possible to ever avoid this disadvantage. The latter has as necessary result that the A-axis must be parallel to the B-axis and that therefore a carriage is necessary also in the B-mechanism. Since the speed of rotation around the A-axis is also very large in the proximity of the point of osculation in relation to the speed of the B-carriage, it will probably be preferable to grind at least the most peripheral part of

the surface only in the direction from the periphery toward the center.

When employing the general sinus mechanism for obtaining an eccentric osculation of the second order, there then results for o = 0.65, at positive radius of curvature of the vertex,

$$= -8^{\circ},0954 \qquad c = 2,7176 \qquad c_0 = 0,283\,38 \qquad c^2 c_0 = 2,0929,$$

and calculations here not included show that the simultaneous centric osculation of the fourth order can be achieved through a minor reduction of o with any desired degree of accuracy. However, the minor difference $\left|c^2 c_0\right| - \left|\frac{\Phi}{3}\right|$ signifies that we cannot decide a priori whether the exact centric osculation of the fourth order is actually more advantageous since this could only be confirmed by a complicated calculation. For the osculation point, $\alpha + \omega = 76.483^{\circ}$ so that consequently the remarks above on the type of machine and grinding are valid without change also in regard to this machine curve.

The maximum value of β is 54.1066°. For the values indicated below, I have calculated the lateral aberration of the rays refracted in the lens with this duplex surface where the direction of the rays was reversed in order to make the calculations less complicated which is without importance for projection at natural magnitude. The values so obtained are listed below:

β	η
10°	0,000 041
20°	0,002 586
30°	−0,002 979
40°	−0,008 696
50°	−0,007 079

Since the optical length on the ray refracted in the point of osculation is the same as on the axis for a surface constructed by this method, the meridian curve of the evolute of the refracted bundle of rays must in this case also have three double-sided cusps and a second ray must also

pass through the axial focal point. In agreement with this, the figures above indicate the respective change of sign.

In order to find out whether it would be possible with other simple machine curves to avoid the large angle $\alpha + \omega$, a series of calculations were carried out from which it resulted that, although the eccentric osculation of the second order can be achieved in various ways, there are always obtained large values either for ω or $\alpha + \omega$ as soon as a simultaneous centric osculation of the fourth order was obtained. Since, among the various duplex surfaces determined in this manner, the one investigated above appears to offer the greatest advantages, this shall not be discussed further here.

As indicated above, this lens was not calculated in view of any sort of practical purpose. Whether it can serve such a purpose, we shall not here decide. Since the principal focus is 1.46713, i.e., insignificantly less than the thickness whereas the diameter exceeds twice the principal focus, it should satisfy high demands on the aperture. However, because of its thickness, it can probably be employed only in those cases where object and image can be sufficiently approached to the respective lens surface so as to permit a low absolute size of the lens. Moreover, the presence of the point of inflection on the meridian curve of the non-spherical surface conditions less favorable reflection on the lens surfaces. It is therefore a moot question whether a bi-hyperbolic lens and/or a combination of two plane-hyperbolic lenses may not be preferable in spite of the double non-spherical surface. Entirely disregarding an eventual achromatization which should be as easily possible in one as in the other case, there is an important circumstance in regard to the sinus condition which speaks very much in favor of the bi-hyperbolic lens.

As well as the sinus condition is understood, the effect capable of being achieved by the satisfaction of the former along a given ray still seems to be misunderstood. The equation valid in a system of revolution for two arbitrary media

$$\Delta n\chi_{II} \sin u = 0$$

signifies, if it is applied to object and image media, that the sagittal magnification coefficient in the projection of an axial point is independent of the inclination of the ray at a constant ratio of the sinus. If this equation is differentiated and divided through the fundamental equation

$$\Delta n\chi_I du = 0$$

also valid for two arbitrary media, there then results

$$\Delta\left(\frac{\sin u}{\chi_I}\frac{d\chi_{II}}{du} + \frac{\chi_{II}}{\chi_I}\cos u\right) = 0,$$

from which we obtain, by employing the abbreviated designations

$$u = u_1 \qquad u' = u'_m \qquad \chi_1 = \frac{\chi'_{Im}}{\chi_{I1}} \qquad \chi_2 = \frac{\chi'_{IIm}}{\chi_{II1}}$$

for m surfaces, the equation

$$\operatorname{tg} u' \frac{d\chi_2}{du'} = \frac{\chi_1 \cos u}{\cos u'} - \chi_2$$

Since $\frac{\chi_1 \cos u}{\cos u'}$ represents the actual tangential magnification in the projection of axial-perpendicular planes into each other for the immediate proximity of a finite inclined ray passing through the axial image point, the last equation therefore shows that the actual tangential magnification is also independent of the inclination of the ray at a constant ratio of the sinus. However, the results are valid only on the assumption that the aberration is eliminated and the sinus condition satisfied along each ray. The effect can be expressed most simply by saying that, in the projection of an infinitely small axial-perpendicular object surface on an axial-perpendicular plane by employing an infinitely small diaphragm, the image

is independent of any arbitrary eccentricity of the diaphragm or, in other words, the aberration is eliminated along each ray not only for the axial point but also for a point located in the same axial-perpendicular plane infinitely close to the axis where this last point is projected in the same axial-perpendicular plane as the axial point.

However, what do we accomplish by satisfying the sinus condition for a given ray with eliminated aberration? As will be immediately apparent from the above equations, we gain by this only that the sagittal magnification along this ray is the same as along the axis whereas the satisfaction of another condition is necessary in order to achieve the same result for the tangential magnification. If consequently an infinitely small diaphragm is decentered so that the respective ray passes through the center of the former, there is then projected an infinitely small axial-perpendicular surface anamorphotically on the axial-perpendicular plane and, for an object point infinitely close to the axis, the aberration is eliminated only along those rays intersecting that line which is perpendicular in the diaphragm center to the plane passing through the latter and the axis so that these rays therefore constitute an infinitely small part of the infinitely thin bundle of rays delimited through the diaphragm.

It follows from this that the mere indication of the sinus ratios corresponding to the various inclinations of the ray is not suitable, as seems to be the general viewpoint, for giving a comprehension of the effect of satisfying the sinus condition along a given ray. In order to derive this comprehension, we must also know the coefficients of the tangential magnification corresponding to the various inclinations of ray for axial-perpendicular object and image planes. If the paraxial magnification coefficient is indicated briefly by χ, we must then determine, in accordance with the

various inclinations of ray, not only the value of $\frac{\chi_2}{\chi} = -\frac{n \sin u}{n' \sin u'}$, but also that of $\frac{\chi_1 \cos u}{\chi \cos u'}$, in order to be able to judge the effect of satisfying the sinus condition for the ray passing through the periphery of the lens. These values have been calculated by me, by employing the exact aberration-eliminating surface, for those inclinations of ray for which the coordinates of the surface points given above were determined, and have utilized these values for the construction of the curves in fig. 8. The inclinations of

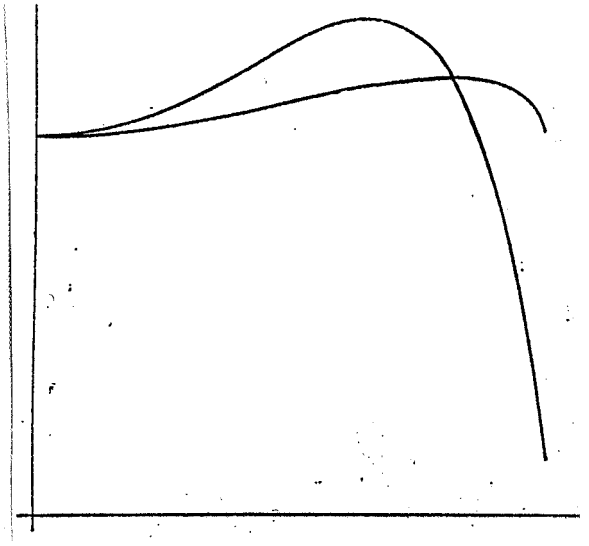

Fig. 8

ray listed in the table above are plotted as abscissae and the corresponding magnification ratios as ordinates. The flatter curve represents the sagittal and the steeply decreasing curve the tangential magnification ratio on the basis of the above indication, and the ordinates of the end points of the former therefore represent the unit of the ordinate scale.

If the sinus condition were satisfied along each ray, the two curves would then coincide in a straight line parallel to the axis of the abscissa. However, in the cases where, as is the case here, this condition is satisfied only for a given inclination of ray, we know a priori nothing on the ordinate of the curve of the tangential magnification ratio corresponding to the former. Inversely, it follows from the last of the equations given

above that the two curves intersect in that point where the tangent of the curve of the sagittal magnification ratio is parallel to the axis of the abscissa. Consequently, there must also exist, if the sinus condition is satisfied for two different rays, a second point of intersection of the two curves. If we let the two rays approach each other infinitely closely, then the common ordinate in the second point of intersection has the value of one. Only when this is the case, can the purpose attempted through satisfaction of the sinus condition be considered as achieved.

It now follows initially, and specifically for the present case, that it is better to satisfy the sinus condition for the peripheral ray than for an intermediate ray. Corresponding to the peripheral ray, the tangential magnification on the axial-perpendicular plane is in any event only about 1/7 of the sagittal magnification which corresponds to a high degree of anamorphotic projection by an infinitely small diaphragm along the margin ["randstehend"]. Moreover, the unfavorable trace of the curve for the tangential magnification seems to very much indicate that the utilization of two non-spherical surfaces will be advantageous where, because of symmetry, the sinus condition is eliminated along each ray. At least, such a lens should be preferable where one projection is concerned. However, if we are confronted only by the task of making all rays issuing from a small light source pass through a narrow aperture, the lens with one non-spherical surface will be able to hold its own, unless mirror images and required thickness of lens produce disadvantages.

Finally, let us merely stress in regard to the aberration-eliminating surfaces that such a surface does not necessarily need to represent the first or last surface of the system but can be located at any desired locus but that then the simple construction utilized here must be replaced by very complicated calculations.

Image-Planating Duplex Surfaces. If, in a given optical system of revolution in which only the locus of the vertex and the radius of curvature of the vertex is prescribed for its last surface, an optimum satisfactory projection of a certain axial-perpendicular plane on an also axial-perpendicular plane is required with a narrow diaphragm in the prescribed locus, this objective can in many cases be achieved by giving the last surface a suitable form. If this form of the surface accomplishes that the two image planes corresponding to the object plane intersect in the axial-perpendicular plane passing through the axial image point, the surface will then be designated as anastigmatically image-planating. If the ratio of the axial distance of the anastigmatic image point to the axial distance of the object point is equal to the axial magnification coefficient, then the non-spherical surface is orthoscopically and anastigmatically image-planating. However, if we achieve for the prescribed inclination of ray only that the sagittal image surface and the axial-perpendicular image plane intersect, the surface will even then be designated as image-planating, provided the tangential image surface intersects this plane at some other point.

It should be noted in regard to these definitions that, like an aberration-eliminating surface, an image-planating surface can also occupy any desired place in the optical system but that I have found a simple method of construction only for the case where this surface is the last and/or first surface of the system. It should also be kept in mind that a surface image-planating in accordance with the above definition, for example orthoscopically and anastigmatically, is not necessarily applicable in practice. Since the definition takes into consideration only a certain inclination of ray, it is therefore conceivable that values not possible in practice are obtained

for small inclinations of ray where, for instance, the meridian curve of the image surfaces may even have infinitely distant points. After calculation of an image-planating surface, we must therefore examine in each case the practical applicability of the latter through the behavior of the image surfaces between the ray forming the basis of calculation and the axis. It will finally also be apparent that, on the basis of these definitions, an image planating surface is characterized, depending on whether the image planation is or is not anastigmatic, by the respective magnitudes determining eccentric osculation of the second and/or first order.

These magnitudes can be found through a geometrical construction in the following manner. In fig. 9, let A be the vertex of the last surface, AB = the axis, CD = the incident principal ray which intersects the axis in the point B, and d be the sagittal focal point corresponding to the given object point in the next to the last medium. In addition, let E be the point conjugate to the axial object point in the last medium and F be the point in the axial-perpendicular plane passing through E in which the given extra-axial object point in the last medium is to be projected. The distance FE is therefore determined in the usual manner through the condition of orthoscopy but is selected freely when we must forego orthoscopy. We then draw the line DFG which intersects the axis at G, plot the normal GH from G on the ray CP and draw a circle with a radius GK = $\frac{n}{n'}$ · GH with G

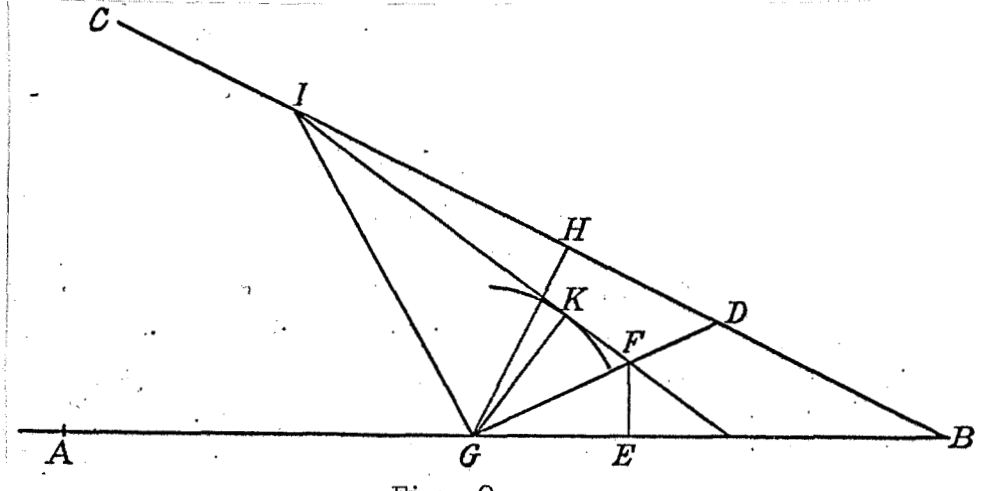

Fig. 9

as its center. The point of intersection I of the tangent of the circle passing through F with the ray CB is the desired surface point and IG the respective surface normal. That the refracted ray will pass through F follows from the construction where the angle GIH and/or GIK represent angles of incidence and/or refraction and it also follows from the construction that F is the sagittal focal point in the last medium because this point must be located on the line DG. Since the sagittal projection is independent of the tangential radius of curvature of the surface at the point I, the non-spherical surface acts in regard to it like a sphere with the radius GI. Finally -- by trigonometry, however -- the tangential radius of curvature at the point I is determined by the condition that the tangential focal point corresponding to the given object point in the last medium shall also be located in the F. If special reasons make it necessary to forego anastigmatic image-planation, the value of the tangential radius of curvature can be varied in order to obtain the optimum tangential image surface by experimentation.

This construction furnishes the trigonometric formulas in which we shall designate, for A as starting point, the coordinates of the points G and F by a b and/or a'b' the radius of circle GK with e and the angle BGD with o and employ otherwise the usual designations utilized for recalculation. In addition to the radius of curvature of the vertex, the indices of refraction and the characteristics of the incident ray, we therefore know a'b' as well as q - ζ and q - τ. We initially have

$$a = s - (q - \varsigma)\cos u \qquad b = (q - \varsigma)\sin u$$

and then obtain M and o by means of the equation

$$\frac{b}{a - M} = \frac{b'}{a' - M} = \operatorname{tg} o,$$

from which results e u' and s' out of the equation

$$e = \frac{n(s - M)\sin u}{n'} = \frac{b' \sin (o + u')}{\sin o} = (s' - M)\sin u'$$

After we have then determined i' and N by utilizing the relations

$$\operatorname{tg} i' = \frac{n \sin (u' - u)}{n' - n \cos (u' - u)} \qquad N = \frac{e}{\sin i'}$$

we also know φ and obtain in the usual manner ρ_1. In the case of $\zeta = \infty$ and $a' = \infty$, the same system of formulas can also be employed with the corresponding easily effected modification. In the former case, the angle IDG = 0 and consequently o = -u and, in the latter, the angle IFG = 0 and consequently o = -u' where u' is determined by the condition of orthoscopy and/or selected freely so that o is known a priori in both cases.

As an example of the application of this method, we shall calculate an orthoscopically and anastigmatically image-planating duplex surface on the supposition of a simple lens with an anterior stop and infinitely distant object. In order to find out first how a plane-convex lens turned with the plane surface to the light, behaves under these circumstances, let us base our calculation on the lens characterized by the scheme $\rho_1 (d/n) \rho_2$ through the numerical values ∞ (0.5/1.53) - 1 with an anterior stop, the center of which is $s_1 = -0.25$ at the inclination of ray $u' = 59^\circ$. Here a' = 1.88679 and the condition of orthoscopy produces b' = - 3.1405 which results, on the basis of the calculation just indicated, in the values

$$M = -2{,}7565 \qquad N = -2{,}2550 \qquad \varphi = 10^\circ{,}0066 \qquad \rho_1 = -3{,}6925$$

A brief reflection shows us that no practically applicable lens can be constructed with these values. Since $\left|\frac{N}{\cos \varphi}\right| < |M|$, the tangent of the meridian curve erected in the respective surface point intersects the axis in a point located on the concave side of the surface from which follows that the meridian curve must have two points of inflection between the respective point and the vertex. For those rays which enter the non-spherical surface at

the points of inflection, the tangential image point lies in infinity, however. Because of these points of inflection, the corresponding duplex surface is therefore practically inapplicable although it can be produced by the utilization of compound machines or special cylinders. Since a reduction of the angle of inclination to 44° will not eliminate this defect as further calculation shows, we have no other choice than either to forego orthoscopy or else to attempt deflection of the lens. Calculations continued in the former direction for the greater inclination of ray indicated above show that the defect is not yet eliminated at $b' = -2.5$ whereas, at $b' = -2$ there occurs the opposite effect $|N| > |M|$. Among the intermediate values, there are several which permit a simple duplex curve as machine curve, e.g., at $b' = -2.25$:

$$M = -1{,}439\,88 \qquad N = -1{,}409\,35 \qquad \varphi = 26^\circ{,}6565 \qquad \rho_1 = -1{,}466\,56.$$

At $o = 0.25$, these values result for the tangential mechanism in the machine constants

$$c_0 = 0{,}067\,396 \qquad c = 6{,}443\,34 \qquad \omega = -43^\circ{,}100,$$

which are mechanically very satisfactory in spite of the unusually high value of c since $\alpha + \omega$ does not reach 15° in accordance with the given point. That this makes it possible to achieve anastigmatic image-planation by means of the plane-convex lens while foregoing orthoscopy is hardly subject to doubt. However, since there would be no practical purpose in examining the image surfaces of the various possible lens forms and to compare them with each other which would require extremely complicated calculations, it will be sufficient here to have indicated the numerical values above as an example of the applicability of the method.

In the attempts for the deflection of the lens, we may now consider the objective of not only satisfying the condition of orthoscopy but of simultaneously making possible also the application of the mechanically

more advantageous sinus mechanism. These attempts were turned over by me to Mr. B. Lindblad who worked for some time in my laboratory. On the assumption of a concave anterior surface, diaphragm distance and lens thickness were determined as ¼ and/or 1/5 of the radius of curvature of this surface and the radius of the second surface varied experimentally by basing the calculations on an angle of inclination of $u_1 = -36.48^\circ$. This produced a favorable lens form - 1(0.2/1.53) - 0.255 with which we obtained, at $s_1 = 0.25$ and $u_1 = -36.48^\circ$

$$M = -0{,}324\,72 \qquad N = -0{,}315\,41 \qquad \varphi = -46^\circ{,}418 \qquad \rho_1 = -0{,}390\,90$$

At o = 0.25 and $\varphi = 0.255$, these values furnished for the sinus mechanism

$$c_0 = 0{,}069\,64 \qquad c = 2{,}1469 \qquad \omega = -50^\circ{,}205,$$

and these machine constants must be regarded as very favorable. At higher values of o, there result numerically greater values of ω and the latter reached amounts of -51.429° and/or -58.087° at o = 1 and/or o = 2. The possibility of influencing the form of the surface through variation of o is therefore restricted a priori and is further limited by the fact that the product $c^2 c_0$ cannot receive a value greater by more than 4%.

Because of the more uniform distribution of the eccentric action on the A- and B-mechanism, the parallel curve determined by o = 1 was selected as machine curve which has the machine constants,

$$c_0 = 0{,}11199 \qquad c = 1{,}7268 \qquad \omega = -51^\circ{,}429$$

for $\rho = 0.255$. By applying these constants, we first determined those points of the duplex surface which correspond to the values $\beta = 5^\circ, 10^\circ, \ldots 60^\circ$ after which those rays were determined which pass through the diaphragm center in the first medium and intersect the duplex surface in these points after refraction in the first surface and we finally calculated the focal points on these rays. Fig. 10 shows the trace of the meridian curves

of the image surfaces constructed in this manner. (The less curved trace represents the sagittal image surface.) The coordinates of the curve points calculated are grouped in the table following below where the rays are identified by the determined values of u_1 and ξ, η, and/or $\xi_{,,} \eta_{,,}$ represent the coordinates of the tangential and/or sagittal focal points if the starting point is shifted to the axial image point.

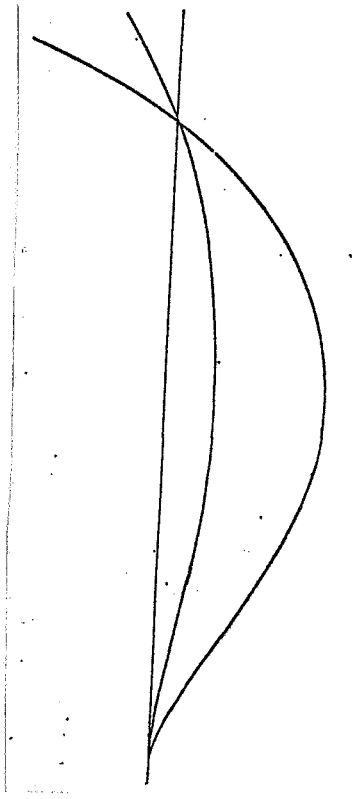

Fig. 10

In addition to the coordinates of the focal points, the table contains in the last column the numerical values Q which afford a review of the so-called zones of distortion. If η represents the ordinate of the point of intersection of a ray refracted in the lens with the axial-perpendicular plane passing through the axial focal point and if D is the power of refraction of the lens, i.e., the reciprocal value of the principal focal length of the latter, then $\eta = -\frac{\operatorname{tg} u_1}{D}$ is the condition of absence of distortion. However, since the latter is satisfied only for a given ray, the number $Q = -\eta D \cdot \cot u_1$ furnishes for the other rays the ratio of the real ordinate to the ordinate which corresponds to complete absence of distortion. At $u_1 = -36.48^\circ$, ξ, and $\xi_{,,}$ pass through zero and Q passes through one whereas $\eta_{,} = \eta_{,,} = 0.4369$. The principal focal length of the lens is 0.5909.

Table
for evaluating the efficiency of an orthoscopically and anastigmatically image-planating duplex surface.

u_1	ξ_I	η_I	ξ_{II}	η_{II}	Q
− 3°,2103	0,0066	0,0333	0,0019	0,0332	1,0023
− 6°,4618	0,0247	0,0684	0,0070	0,0677	1,0082
− 9°,7786	0,0490	0,1063	0,0130	0,1043	1,0168
− 13°,1894	0,0742	0,1480	0,0225	0,1439	1,0260
− 16°,6743	0,0939	0,1927	0,0281	0,1857	1,0326
− 20°,2456	0,1049	0,2401	0,0322	0,2303	1,0367
− 23°,8801	0,1028	0,2882	0,0322	0,2764	1,0393
− 27°,5683	0,0886	0,3360	0,0293	0,3242	1,0320
− 31°,2856	0,0601	0,3811	0,0207	0,3719	1,0224
− 35°,0253	0,0183	0,4220	0,0068	0,4190	1,0073
− 38°,7714	− 0,0337	0,4580	− 0,0125	0,4644	0,9866
− 42°,5093	− 0,0970	0,4871	− 0,0372	0,5062	0,9601

These values only represent the result of the first attempt of deflection and could probably be improved through further experimentation. If the unit is made equal to 10 cm, the lens is then suitable for an eyepiece because the diaphragm center utilized in the calculation can coincide with the center of revolution of the eye. At a focal length of about 6 cm, the object plane could have a diameter of at least 9 cm and the eye piece would therefore give a full-scale reproduction of corresponding wide-angle photographs. However, whether the construction of such an eye piece would be advantageous without achromatization shall not be decided here.

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Among the possibilities of application of the duplex surfaces, let us here briefly call attention to the non-focal lenses. A thin lens with infinitely distant principal focuses and with spherical surfaces has a very minor action on an optical system. However, if one surface is non-spherical, then only the action on the paraxial projection remains irrevelant whereas -- depending on the locus of the diaphragm -- the aberration on the axis or the properties of the extra-axial projection or both are changed.

Such a lens can be incorporated without difficulty as a non-focal complementary lens in most optical instruments and offers an additional means of correction by deflection. The characteristics of optical projection here require in most of the optical instruments utilized so far a lens which is thicker at the periphery than in the center so that the latter can really be thin as far as the paraxial projection is concerned. In cases where a plane is preferable to the spherical surface, we need only utilize a non-focal duplex surface.

Practical preliminary experimentation on small machines had been completed by me in the first half of 1914. The findings showed that neither the method of grinding surfaces of the second degree nor the duplex method encounters serious difficulties so that an agreement with a large foreign company had been reached in July 1914 for immediate construction of a duplex machine for actual practical operation which was to be ready within six months. World War I has been the reason why practical technical experience has not kept step with theoretical developments.

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